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**Cosmological Redshift and
Gravitational Potential in a Spatially
Flat Universe.**

**Submitted for the degree of
Master of Philosophy
The Open University
Physics and Astronomy Department**

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Abstract

We take a new look at an old concept. Textbook descriptions of cosmological redshift are examined. In particular, explanations that treat cosmological redshift as a continual loss of energy as the radiation propagates are found to be hard to reconcile with Quantum Theory. A new treatment of the subject is presented that compares cosmological redshift with gravitational redshift as demonstrated by the “classic” Pound Rebka experiment. Cosmological redshift in a spatially flat universe, is presented as being due to a rising “global” gravitational potential. The evidence for the Universe being essentially spatially flat is reviewed and found to be compelling. The relationship between the scale factor of the universe and the potential is found and a modified version of the Robertson Walker metric is derived with the components of the metric tensor being expressed in terms of the global potential. A simple treatment of the quantum electromagnetic field is shown to be compatible with the notion of cosmological redshift, if the time coordinate of the background space-time in which the theory is expressed, is that of the new modified Robertson Walker metric. Finally the possibility of building an apparatus that would directly detect the rising potential is examined. For such an apparatus it is shown that in a non-accelerating universe the effects of the rising potential are unobservable in local experiments, but that there may be observable effects if the universe is accelerating.

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Chapter 1

Introduction

The initial research project and how it became a study of cosmological redshift. The original 1997 proposal for this project envisaged a final document covering the following topics:

- 1 A review of how gravitational waves arise in General Relativity theory.
- 2 A review the theory of generation and detection of gravitational waves.
- 3 A review of current and proposed gravitational wave detectors.
- 4 A review of “relic gravitational waves” from the “Big Bang”.
- 5 The development of a model for the “lensing” of relic wave background by massive objects.
- 6 An investigation of the possibility of detecting lensing of relic waves with proposed detectors.

A significant amount of work was done to develop the necessary mathematical skills and knowledge of General Relativity and some chapters were written out in draft form. Many review articles were found on the latest interferometer gravity wave detectors (GEO 600, LIGO, VIGO) and on the proposed space based detector (LISA). The key starting point paper was “The detection of gravitational waves” by B. F. Schutz, published in “Relativistic Gravitation and Gravitational Radiation” [39]. This paper gives an introduction to the nature of gravitational waves, the theory and practice of detection and an introduction to likely sources. In the same publication is

a paper by B. Allen “The stochastic gravity-wave background: sources and detection” [40]. This paper introduces the idea of a gravitational analogue of the cosmic microwave background. This gravitational radiation, arising at the earliest stages of the evolution of the universe, would have been in thermal equilibrium with the matter and other radiation of the time. Arguments are presented for a relic field having a black body spectrum at a temperature of 0.9K now. It was in looking at explanations of cosmological redshift in numerous textbooks on general relativity and cosmology that the author became interested in a better “physical” (or at least more satisfying to the author) explanation of redshift based on the idea that it might be analogous to the gravitational redshift as measured by Pound and Rebka. The nature of the whole exercise was thus taken over by the pursuit of this treatment of cosmological redshift.

Radiation from distant parts of the Universe arrives at us “cosmologically redshifted”. Many textbooks attempt to explain this with the help of various models such as expanding sheets or waves in mirrored boxes. These models arrive at the expected scaling of wavelengths with increasing scale factor of the Universe, whilst treating this expansion as a process that occurs while the radiation is propagating from emitter to observer. Quantum mechanical arguments against such a process are presented and this investigation attempts to explain how the cosmological redshift might be understood in a way that arises naturally from the metric in a universe described by a suitably modified, spatially flat Robertson Walker metric. The investigation associates gravitational potentials at the time of emission and observation with the metric components of a modified version of the spatially flat Robertson Walker metric and explains cosmological redshift as the result of time dilation (and therefore energy difference) due to the difference in potential at the time of emission and observation. In this way, the redshift is seen as nothing more mysterious than say a system of particles with total energy E in one reference frame having an energy $E' \neq E$ in another, and there is no process occurring as the radiation propagates. It also looks at the effect of expansion on the paths of massive particles with non zero initial velocity

with respect to the average local matter field.

Setting the scene. In chapters 2, 3, 4 and 5 the investigation starts by looking at the implications of an assumption of isotropy in the Universe, leading to Hubble's Law and the notion of "Cosmic time". It then shows how the Robertson Walker (henceforth RW) metric comes about and discusses expansion, isotropy and cosmic time in terms of the RW metric. It demonstrates that the RW metric is consistent with General Relativity and looks at the "dynamics" of the universe. It shows that "co-moving" observer world lines are geodesics in the space-time described by the RW metric and that the physics of cosmological redshift as applied to massive particles is already contained within the RW description of the universe.

Justification for the concentration of the spatially flat form of the Robertson Walker metric. In chapter 6, the evidence for the spatially flat universe being a "reasonable" model for the actual Universe is explored. It includes evidence from the COBE satellite and many more recent observations that support the idea that the universe is spatially flat, at least to a good approximation. It looks at the latest evidence for a non zero cosmological constant and for evidence of acceleration.

Review of text book descriptions of cosmological redshift. Next, in chapter 7, comes a review and criticism of the current literature on the subject of cosmological redshift. A simple quantum mechanical argument due to Zel'dovich is presented that implies that photons cannot undergo any interactions during their flight from distant sources to an observer and the textbook descriptions are examined with this argument in mind.

Gravitational potential and metric components in the spatially flat RW metric universe. The investigation moves on in chapter 8, to derive a "global" time varying potential term. It shows how the metric tensor components can be expressed in terms of this potential and shows that the physics of cosmological redshift as applied to massless radiation arises from this modified version of the RW model of the universe. It demonstrates

that the new metric is consistent with General Relativity and examines its consequences. The Relationship between the cosmic potential and matter and the cosmological constant is examined and simple expressions for the Hubble term in terms of the global potential, and the relationships between the potential and the density and cosmological constant are derived.

Killing vectors, symmetry and energy conservation. Chapter 9 first looks at Killing vectors and symmetry and their relationship to conservation laws. In the light of this, it then looks at the RW metric and the modified RW metric and discusses energy conservation for photons from distant sources in an expanding universe.

Quantum field theory (QFT) in an expanding universe. Chapter 10 looks at the type of quantum field theory that assumes a background space-time (often not explicitly stated) and shows that such a structure of QFT can be used to describe the electromagnetic (EM) fields in an expanding universe if the time coordinate in a Robertson Walker model, is modified in the way shown. The consequences of having non-zero vacuum energy are discussed.

Conclusions and proposals for further work. In chapter 11, the results are summarized and some suggestions made for further work.

Chapter 2

The assumption of isotropy and its implications

2.1 Expansion (or contraction)

We expect and will assume that the Universe appears isotropic to all observers. For example the cosmic microwave background intensity as seen by all observers will be the same in all directions. In particular, for an observer surrounded by an isotropic matter distribution, the mass density can be a function of distance **only**. There can be no preferred direction for density or other attributes such as the velocity strain tensor $\partial v_i / \partial x_j$. See [1] page 65. The velocity v_i is the i^{th} component of the velocity of the average matter in a region some distance away from an observer in any (all) direction.

Letting i and j take the value 1, 2 and 3 such that $v_1 = v_x$, $v_2 = v_y$ and $v_3 = v_z$ then isotropy requires

$$\frac{\partial v_x}{\partial x} = \frac{\partial v_y}{\partial y} = \frac{\partial v_z}{\partial z}$$

and $\partial v_i / \partial x_j = 0$ for all $i \neq j$

Now if $\mathbf{v}_r = \mathbf{v}_x + \mathbf{v}_y + \mathbf{v}_z = f(\mathbf{r})$ ($\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$) a function of the magnitude of \mathbf{r} only (bold type indicates a vector), where \mathbf{v}_r is the increase

in proper distance between observer and observed per unit observer proper time, then isotropy requires $\mathbf{v}_r = 0$ when $\mathbf{r} = 0$

If the above results taken together, apply to any arbitrary observer in the universe then they imply $\mathbf{v}_r = H\mathbf{r}$ where H is a scalar known as Hubble's constant. That is, the only allowed velocity field is a radial expansion or contraction with velocity proportional to distance. *Note - the Hubble term H is a function of time, $H(t)$.*

What we now need is a space-time metric, consistent with General Relativity, that describes a homogeneous, isotropic universe and produces expansion or contraction that follows the $\mathbf{v}_r = H\mathbf{r}$ law. This will turn out to be the Robertson Walker metric.

2.2 Cosmic time

In a homogeneous and isotropic universe, where the above velocity law holds, we would expect the matter density to decrease (or increase) with time. A group of observers, each one at rest with respect to its average local matter field (or CMBR) and all equidistant but in random directions from another observer, also at rest with respect to its local average matter field, would see the density of matter in the vicinity of the observer, vary with time (or equivalently see the CMBR temperature vary with time). When it reached an agreed value, the observers could send a signal to the central observer who would note the coincidence arrival of all the signals and might consider this evidence for a universal standard of time upon which all observers could agree.

Obviously, this assumes a fine even distribution of matter. One observer near the event horizon of a black hole would not agree with the standard of time of all the other observers.

This then is the concept of cosmic time. The metric we seek should show how this arises in an evenly distributed isotropic homogeneous universe.

2.3 Co-moving observers and geodesics

We will see that in the Robertson Walker metric description of an expanding universe, we can assign fixed coordinate values to all observers who are at rest with respect to their average local matter field. Such observers are said to be co-moving and we should be able to demonstrate that the world lines of co-moving observers are geodesics in the space-time described by the RW metric.

Chapter 3

The Robertson Walker metric

3.1 A hypersphere in Euclidean 4 space

Using our notion of cosmic time, we allow ourselves to take a 3 dimensional slice of space-time at a particular value of cosmic time $t = \tau$. In a homogeneous and isotropic universe, the Gaussian curvature $K(\tau)$ of all the geodesics on the hypersurface should be the same value (if this were not so then the hypersurface geometry wouldn't be isotropic and homogeneous). So at time $t = \tau$ we can write

$$K(\tau) = k/R^2(\tau)$$

(see [2] page 148) where k gives the sign of the curvature (+1, 0, -1) and $1/R^2(\tau)$ the magnitude. Allowing time to vary we get

$$K(t) = k/R^2(t)$$

and if we set $k = 1$ then the space is a hypersphere the squared radius of which ($R^2(t)$) is a function of time.

We can imagine this hypersphere embedded in a 4 dimensional Euclidean space in which the square of the radius at time $t = \tau$ is given, in terms of Cartesian coordinates thus

$$x^2 + y^2 + z^2 + w^2 = R^2(\tau).$$

Transforming the Cartesian system of x, y, z into spherical polars r, θ, ϕ , would give

$$r^2 + w^2 = R^2(\tau)$$

where at time $t = \tau$, $R^2(\tau) = a$, a constant. Thus $r^2 + w^2 = a$. Differentiating with respect to some parameter u on the hypersphere, we can see how r and w vary jointly over the hypersphere, thus

$$\frac{da}{du} = 0 = 2r \frac{dr}{du} + 2w \frac{dw}{du},$$

so

$$r dr = -w dw$$

or

$$r^2 dr^2 = w^2 dw^2.$$

Rearranging we get

$$dw^2 = r^2 dr^2 / w^2$$

but

$$w^2 + r^2 = R^2(\tau)$$

or

$$w^2 = R^2(\tau) - r^2.$$

Thus

$$dw^2 = r^2 dr^2 / (R^2(\tau) - r^2)$$

or as time varies

$$dw^2 = r^2 dr^2 / (R^2(t) - r^2).$$

3.2 A metric on the hypersphere

The square of the elemental length on the hypersphere dl^2 is

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 + dw^2.$$

Putting $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and using the result $dw^2 = r^2 dr^2 / (R^2(t) - r^2)$ gives

$$\begin{aligned} dl^2 &= dr^2 + \frac{r^2 dr^2}{R^2(t) - r^2} + r^2 d\Omega^2 \\ dl^2 &= \frac{dr^2(R^2(t) - r^2)}{R^2(t) - r^2} + \frac{r^2 dr^2}{R^2(t) - r^2} + r^2 d\Omega^2 \\ dl^2 &= \frac{R^2(t) dr^2}{R^2(t) - r^2} + r^2 d\Omega^2. \end{aligned}$$

We can see from the above that equatorial shifts of $d\theta$ (that is one in which $\phi = d\phi = 0$ and thus $d\Omega = d\theta$) give displacements $r d\theta$. Non equatorial shifts can always be transformed into equatorial ones by coordinate transform that give $\phi' = 0$ and thus $d\Omega' = d\theta'$. A change in radial distance dr gives a displacement $R(t) dr / \sqrt{R^2(t) - r^2}$. These results hold at all points on the hypersphere and thus it is homogeneous, isotropic and curved.

3.3 Co-moving coordinates

If we now make the substitution $\sigma = r/R(t)$ then we can rewrite the metric

$$dl^2 = R^2(t) \left(\frac{d\sigma^2}{1 - \sigma^2} + \sigma^2 d\Omega^2 \right)$$

σ and Ω now form a coordinate system that expands (or contracts) with the scale factor $R(t)$ of the hypersphere. The coordinates of points on the hypersphere remain constant as $R(t)$ varies. Such coordinates are called co-moving.

3.4 A locally Minkowski space-time metric

For slowly varying $R(t)$, a metric can be constructed that is locally Minkowski and has the space part of the previous subsection, thus

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left(\frac{d\sigma^2}{1 - \sigma^2} + \sigma^2 d\Omega^2 \right)$$

t, σ and Ω (combining θ and ϕ) form a co-moving set of space-time coordinates for an expanding (or contracting) universe that is isotropic, homogeneous and has positive curvature everywhere.

3.5 Co-moving observers and cosmic time

The metric equation derived above is

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left(\frac{d\sigma^2}{1 - \sigma^2} + \sigma^2 d\Omega^2 \right).$$

Co-moving observers are those whose spatial coordinates do not vary with time (these are the observers we expect to be at rest with respect to their local matter field). For such observers $dr/dt = d\Omega/dt = 0$ and thus $dr^2 = d\Omega^2 = 0$ so the metric equation reduces to

$$c^2 d\tau^2 = c^2 dt^2$$

therefore $\tau = t$.

From this we see that the proper time between two events in this space-time is the same as the time measured by any co-moving observer between the same two events. Thus this metric supports the notion of cosmic time upon which all observers who are at rest with respect to their local matter field agree, provided we make the identification between co-moving observers and observers who are at rest with respect to their local matter field.

3.6 The Robertson Walker metric

In deriving the metric equation

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left(\frac{d\sigma^2}{1 - \sigma^2} + \sigma^2 d\Omega^2 \right)$$

we started by considering a hypersphere of radius $R^2(t)$ and curvature $K(t) = k/R^2(t)$ where we set $k = 1$ for positive curvature.

We leave the above metric equation unaltered if $k = 1$ by incorporating k as shown below.

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left(\frac{d\sigma^2}{1 - k\sigma^2} + \sigma^2 d\Omega^2 \right).$$

Letting $k = 0$ gives

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) (d\sigma^2 + \sigma^2 d\Omega^2)$$

which is clearly spatially flat (Euclidean) and the $k = -1$ describes the negative curvature case. This metric equation, incorporating k is the Robertson Walker metric.

3.7 Distance in the Robertson Walker metric

We can arrange our coordinate system such that the (co-moving) spatial coordinates (r, θ, ϕ) of two objects (galaxies perhaps) are $(0, 0, 0)$ and $(\sigma, 0, 0)$ respectively. From the spatial part of the metric equation we can see that the distance s between them is

$$s = R(t) \int_0^\sigma \frac{dr}{\sqrt{1 - kr^2}}.$$

The solutions are (see [2] page 150):

For $k = 1$, $s = R(t) \sin^{-1} \sigma$, representing a closed positive curvature universe.

For $k = 0$, $s = R(t) \sigma$, representing an open spatially flat universe.

For $k = -1$, $s = R(t) \sinh^{-1} \sigma$, representing an open universe of hyperbolic geometry (negative curvature).

It is worth noting that these distances are proper distances at a fixed local (observer's proper) time t .

3.8 Hubble's law and the Robertson Walker metric

We can see that the $k = 0$ case reproduces the velocity law we derived in chapter 2 section 1.

Since the proper distance between one galaxy and another, $d = R(t)\sigma$ then the velocity (defined as the rate of change of proper distance with coordinate time) of one with respect to the other is v where $v = \dot{R}(t)\sigma$ but $\sigma = r/R(t)$. So $v = (\dot{R}(t)/R(t))r$ and identifying $H = \dot{R}(t)/R(t)$ we arrive at the velocity law

$$v = Hr.$$

Chapter 4

Is the Robertson Walker metric consistent with General Relativity ?

4.1 The tasks ahead

We have two main tasks in this section. The first is to show that co-moving observers are in free fall in this metric. The second is to show that the space-time described by this metric comes about (from the point of view of General Relativity) from a uniform distribution of matter in which the geodesics followed by local matter in free fall are those of co-moving observers.

4.2 Co-moving observers and geodesics.

4.2.1 Covariant differentiation and metric connections.

If we look at say the four momentum of a test particle being acted on by some force which might be a combination of gravitational and non gravitational force, then we expect that as it follows a path, s , in curved space-time, the component of momentum in the μ direction, q^μ will change. Some of this change will be due to the changes in the coordinate frame along the path s and is not representative of any physical process occurring (such as

acceleration) along the path. If the changes due to the coordinate frame along a path element Δs are written as δq^μ and the total change is Δq^μ then the change that IS due to a physical process is $\Delta q^\mu - \delta q^\mu$.

If we divide the above quantity by the element of the path length Δs and take the limit as $\Delta s \rightarrow 0$ then we arrive at the covariant derivative of q^μ which in the notation used by Kenyon ([2] Page 60) we write as

$$\frac{Dq^\mu}{Ds} = \lim_{\Delta s \rightarrow 0} \frac{\Delta q^\mu - \delta q^\mu}{\Delta s}.$$

One of the fundamental principles of General Relativity, the “strong equivalence principle”, may be expressed as follows: Over a short enough path length, it is always possible to transform to a free fall frame of reference, where space-time is locally flat and the coordinates of Special Relativity may be used. We can use this idea to define an unambiguous method of determining the δq^μ s.

If we take a local vector at some point P , we can transform it into a free fall frame, carry it across the (small) path interval Δs with no change in its coordinates in the free fall frame, to a point P' . If we then transform it back to the original frame, we would find that a vector that for instance had only a component in the σ direction, will in general end up with components in all directions. This process is called “parallel transport”.

If a vector q^μ is parallel transported a distance Δx^ρ in the ρ direction then the changes δq^μ are given by

$$\delta q^\mu = -\Gamma_{\sigma\rho}^\mu q^\sigma \Delta x^\rho.$$

where $\Gamma_{\sigma\rho}^\mu$ gives the change in the μ component of a vector due to its original σ component when transported in the ρ direction a distance Δx^ρ .

These quantities $\Gamma_{\sigma\rho}^\mu$ are known as metric connections and are a special case of affine connections that apply when the space is not necessarily Riemannian.

We can substitute $\delta q^\mu = -\Gamma_{\sigma\rho}^\mu q^\sigma \Delta x^\rho$ into $Dq^\mu/Ds = \lim_{\Delta s \rightarrow 0}(\Delta q^\mu - \delta q^\mu)/\Delta s$. and get

$$\frac{Dq^\mu}{Ds} = \frac{dq^\mu}{ds} + \Gamma_{\sigma\rho}^\mu q^\sigma \frac{dx^\rho}{ds}.$$

The covariant derivative is a tensor and thus equations involving covariant derivatives of vectors (or tensors) and other tensors will hold in all frames. The ordinary derivative dq^μ/ds is not itself a tensor as it is frame dependant. This frame dependence is “corrected for” by the term $\Gamma_{\sigma\rho}^\mu q^\sigma dx^\rho/ds$ and the resultant covariant derivative is what represents real physical change in a system.

Kenyon [2] page 62 shows that for the covector q_μ the covariant derivative is given by

$$\frac{Dq_\mu}{Ds} = \frac{dq_\mu}{ds} - \Gamma_{\sigma\rho}^\mu q^\sigma \frac{dx^\rho}{ds}.$$

and that for an arbitrary second rank tensor $A_{\mu\nu}$ that

$$\frac{DA_{\mu\nu}}{Ds} = \frac{dA_{\mu\nu}}{ds} - \Gamma_{\mu\rho}^\tau A_{\tau\nu} \frac{dx^\rho}{ds} - \Gamma_{\nu\rho}^\tau A_{\mu\tau} \frac{dx^\rho}{ds}.$$

The metric connections contain all the information there is about the space-time and we should therefore find that the connections are functions of the metric coefficients $g_{\mu\nu}$.

From the equation above for the covariant derivative of a second rank tensor we can write (multiplying through by $\partial s/\partial x^\rho$)

$$\frac{Dg_{\mu\nu}}{Dx^\rho} = \frac{dg_{\mu\nu}}{dx^\rho} - \Gamma_{\mu\rho}^\tau g_{\tau\nu} \frac{dx^\rho}{dx^\rho} - \Gamma_{\nu\rho}^\tau g_{\mu\tau} \frac{dx^\rho}{dx^\rho}.$$

In the free fall frame we expect the connections to vanish (as the changes δq^μ vanish) so that

$$\frac{Dg_{\mu\nu}}{Dx^\rho} = \frac{dg_{\mu\nu}}{dx^\rho}$$

and since locally in free fall $g_{\mu\nu} = \eta_{\mu\nu}$ so $dg_{\mu\nu}/dx^\rho = 0$, then

$$\frac{Dg_{\mu\nu}}{Dx^\rho} = 0.$$

This is a tensor equation and therefore holds in all frames so we can write

$$\frac{dg_{\mu\nu}}{dx^\rho} = \Gamma_{\mu\rho}^\tau g_{\tau\nu} + \Gamma_{\nu\rho}^\tau g_{\mu\tau}.$$

Now define

$$\Gamma_{\mu\nu\rho} = g_{\mu\tau}\Gamma_{\nu\rho}^{\tau}$$

we get

$$\frac{dg_{\mu\nu}}{dx^{\rho}} = \Gamma_{\nu\mu\rho} + \Gamma_{\mu\nu\rho}$$

and using the notation (subscript comma notation) $g_{\mu\nu,\rho} = dg_{\mu\nu}/dx^{\rho}$ we get

$$g_{\mu\nu,\rho} = \Gamma_{\nu\mu\rho} + \Gamma_{\mu\nu\rho}$$

from which ([2] Appendix C) arrives at

$$2\Gamma_{\nu\mu\rho} = g_{\mu\nu,\rho} - g_{\rho\mu,\nu} + g_{\nu\rho,\mu}.$$

4.2.2 Metric connections and the geodesic equation

In Special Relativity, the action of a non gravitational force on a test body may be expressed in terms of the force 4-vector and the momentum 4-vector thus:

$$F^{\mu} = \frac{dp^{\mu}}{d\tau}.$$

We can make this generally covariant by writing

$$F^{\mu} = \frac{Dp^{\mu}}{D\tau}.$$

In the absence of any non gravitational force this becomes

$$\frac{Dp^{\mu}}{D\tau} = 0.$$

We have already seen that the covariant derivative of a vector is

$$\frac{Dq^{\mu}}{Ds} = \frac{dq^{\mu}}{ds} - \Gamma_{\sigma\rho}^{\mu}q^{\sigma}\frac{dx^{\rho}}{ds}.$$

multiplying through by $\partial s/\partial\tau$ and substituting p^{μ} for q^{μ} gives

$$\frac{Dp^{\mu}}{D\tau} = \frac{dp^{\mu}}{d\tau} - \Gamma_{\sigma\rho}^{\mu}p^{\sigma}\frac{dx^{\rho}}{d\tau} = 0.$$

Now

$$p^{\mu} = m\frac{dx^{\mu}}{d\tau}$$

so

$$m \frac{d}{d\tau} \frac{dx^\mu}{d\tau} - \Gamma_{\sigma\rho}^\mu m \frac{dx^\sigma}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

or

$$\frac{d^2 x^\mu}{d\tau^2} - \Gamma_{\sigma\rho}^\mu \frac{dx^\sigma}{d\tau} \frac{dx^\rho}{d\tau} = 0.$$

This then is the geodesic equation, where the geodesic is parameterized by the proper time τ along the path. There are other valid ways of parameterizing the path, for instance proper distance, although this would be no good for a co-moving observer, just as proper time would be no good for a light ray, following a null geodesic.

It is interesting to note that the geodesic equation is independent of the mass of the particle and thus the equivalence principle is built into the Riemann space-time.

4.2.3 Co-moving observer

For a co-moving observer, σ, θ, ϕ are all constant so

$$\frac{d\sigma}{d\tau} = \frac{d\theta}{d\tau} = \frac{d\phi}{d\tau} = 0$$

and

$$\frac{d^2\sigma}{d\tau^2} = \frac{d^2\theta}{d\tau^2} = \frac{d^2\phi}{d\tau^2} = 0.$$

Also, since we know that $t = \tau$, $dt/d\tau = 1$ and thus $d^2t/d\tau^2 = 0$, the (set of) geodesic equation(s)

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

reduce to

$$\Gamma_{00}^\mu \frac{dt}{d\tau} \frac{dt}{d\tau} = 0.$$

So all that is required to show that a co-moving observer follows a geodesic path in Robertson Walker space-time is to show that

$$\Gamma_{00}^\mu = 0.$$

The non zero elements of the metric tensor are :

$$g_{00} = 1, \quad g_{11} = \frac{-R^2(t)}{1 - k\sigma^2}, \quad g_{22} = -R^2(t)\sigma^2, \quad g_{33} = -R^2(t)\sigma^2 \sin^2 \theta$$

This being a diagonal matrix and $g^{\nu\rho}g_{\nu\rho} = \delta_\nu^\nu$ (the Kronecker delta) we can write down the elements

$$g^{00} = 1, \quad g^{11} = \frac{k\sigma^2 - 1}{R^2(t)}, \quad g^{22} = -1/R^2(t)\sigma^2, \quad g^{33} = -1/R^2(t)\sigma^2 \sin^2 \theta.$$

We must now calculate the connections in the form $\Gamma_{\mu\nu\rho}$

$$\Gamma_{\nu 00} = g_{0\nu,0} - g_{00,\nu} + g_{\nu 0,0}.$$

For $\nu = t$ we get

$$\Gamma_{000} = g_{00,0} - g_{00,0} + g_{00,0} = 0.$$

For $\nu = \sigma$ we get

$$\Gamma_{100} = g_{01,0} - g_{00,1} + g_{10,0} = 0.$$

For $\nu = \theta$ we get

$$\Gamma_{200} = g_{02,0} - g_{00,2} + g_{20,0} = 0.$$

For $\nu = \phi$ we get

$$\Gamma_{300} = g_{03,0} - g_{00,3} + g_{30,0} = 0.$$

Calculating connections of the form

$$\Gamma_{00}^\nu = g^{\nu\lambda}\Gamma_{\lambda 00}$$

gives $\Gamma_{00}^\nu = 0$ as required as we have seen that $\Gamma_{\lambda 00} = 0$ for all λ . We have now proved that co-moving observers are in free fall in the Robertson Walker space-time.

4.3 Matter in the Robertson Walker space-time

We must now show that the Robertson Walker metric results from the solution of Einstein's field equations when the stress-energy tensor is that of

finely divided evenly distributed matter, where local matter in free fall follows the geodesics of co-moving observers.

A good discussion of the stress-energy tensor of fine uniform dust is found in [3] Chapter 4. The components of the stress-energy tensor are described as:

T^{00}	energy density
T^{0i}	flux of energy across $x^i = \text{constant}$ surface
T^{i0}	flux of i momentum across $t = \text{constant}$ surface
T^{ij}	flux of i momentum across $x^j = \text{constant}$ surface.

In a volume of uniform non expanding dust where the pressure and temperature are zero, the energy density at some point within this volume will be, in a reference frame at rest with respect to the dust, dominated by the rest mass density of the dust. All other terms will be zero.

Thus $T^{00} = \rho = mn$ where m is the average dust particle rest mass and n is the average number of dust particles per unit volume (note: we have set $c = 1$). Therefore,

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the Robertson Walker metric the same dust would be represented by a stress-energy tensor with different elements - there would not be only one non-zero element. However, it is clear that if the dust were expanding such that each particle followed the world line of a co-moving observer then the number of particles in a volume defined by boundaries of fixed **coordinate** value (and **not** fixed length), would be constant. We can therefore define a new density

$$\rho_{RW} = mn_{RW}$$

where m is the average rest mass of a dust particle and n_{RW} is the number of particles in a volume bounded by fixed coordinate values.

The stress-energy tensor in this (Robertson Walker) co-moving coordinate system is

$$T_{RW}^{\mu\nu} = \begin{pmatrix} \rho_{RW} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So simple a form for the stress-energy tensor eases the task of solving Einstein's field equations and arriving at a metric tensor for the resulting space-time.

Note - this is a valid candidate as a stress-energy tensor since it is symmetric i.e. $T^{\alpha\beta} = T^{\beta\alpha}$ and its 4-divergence is obviously zero.

4.4 Solving the field equations for the above stress-energy tensor

Einstein's field equations are :

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta}$$

(in units such that $c = 1$).

Using our stress-energy tensor $T_{RW}^{\alpha\beta}$ we can simplify the problem to

$$G_{RW}^{00} = 8\pi\rho_{RW}$$

which gives

$$R^{00} - g^{00}R/2 = 8\pi\rho_{RW}.$$

The simplest way to proceed now is to take the metric upon which we wish to concentrate, i.e. the Robertson Walker metric and show that it is consistent with the above equation.

We will examine the $k = 0$ case, since recent observational evidence strongly supports this view (see chapter 6) and will assume firstly that the cosmological constant is also zero ($\Lambda = 0$).

The metric we wish to examine is, therefore

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)(d\sigma^2 + \sigma^2 d\Omega^2)$$

The non zero elements of this metric are

$$g_{00} = 1, \quad g_{11} = -R^2(t), \quad g_{22} = -R^2(t)\sigma^2, \quad g_{33} = -R^2(t)\sigma^2 \sin^2 \theta,$$

giving

$$g^{00} = 1, \quad g^{11} = -R^{-2}(t), \quad g^{22} = -1/R^2(t)\sigma^2, \quad g^{33} = -1/R^2(t)\sigma^2 \sin^2 \theta.$$

Firstly we will generate the components of the Riemann tensor

$$2R_{\alpha\beta\gamma\delta} = g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta}.$$

Then calculate the Ricci tensor elements by the contraction

$$R_{\beta\delta} = R_{\beta\alpha\delta}^{\alpha} = g^{\alpha\sigma} R_{\sigma\beta\alpha\delta}.$$

A further contraction yields the Ricci scalar (Note that this is not the same as the scale factor R in the Robertson walker metric)

$$R = R_{\delta}^{\delta} = g^{\delta\beta} R_{\beta\delta}.$$

Similarly, raising the two indices using the metric tensor yields $R^{\beta\delta}$. We can now substitute our expressions for R , R^{00} and g^{00} into the equation

$$R^{00} - g^{00}R/2 = 8\pi\rho_{RW}.$$

To make this task easier we will use Mathematica and the add-on package Mathtensor. The file shown below is a Mathematica file designed to be used by the MathTensor package “Components” to calculate the components of the Einstein, Riemann and Weyl tensors from the metric components supplied in the file (below). See [4] page 130. In this file the coordinates are defined as r , θ , ϕ and t and the metric components in covariant ($g_{\mu\nu}$) form are those of the spatially flat Robertson Walker metric in spherical polar coordinates. In the format expected by the “Components” package indices

preceded by minus signs are covariant and those without are contravariant. Thus $\text{Metricg}[-1,-1]$ represents g_{11} and $\text{Metricg}[1,1]$ represents g^{11} . In the conventions of this package, indices 1, 2 and 3 represent spatial dimensions and 4 represents time. The sign convention is $+++ -$.

Here is the file!

```
(* Compinrw is the file that is used to calculate the Riemann,
Ricci and Einstein tensors
for a spatially flat Robertson Walker metric Universe *)
Dimension = 4
(* CompSimpRules = {} *)
(* CompSimp[a_] := Together[Expand[a/. CompSimpRules]] *)
x/: x[1] = r
x/: x[2] = theta
x/: x[3] = phi
x/: x[4] = t
Metricg/: Metricg[-1,-1] = a[t]^2
Metricg/: Metricg[-2,-1] = 0
Metricg/: Metricg[-3,-1] = 0
Metricg/: Metricg[-4,-1] = 0
Metricg/: Metricg[-2,-2] = (r * a[t])^2
Metricg/: Metricg[-3,-2] = 0
Metricg/: Metricg[-4,-2] = 0
Metricg/: Metricg[-3,-3] = (r * a[t] * Sin[theta])^2
Metricg/: Metricg[-4,-3] = 0
Metricg/: Metricg[-4,-4] = -c
Rmsign = 1;
Rcsign = 1;
CalcEinstein = 1;
CalcRiemann = 1;
CalcWeyl = 1;
(* end of file *)
```


In a Mathematica session (Figures 4.1 & 4.2) we can use the above file to display values of the Einstein tensor thus:

From the Mathematica session in Figure 4.2 we can see that

$$G_{00} = 3\dot{R}^2(t)/R^2(t)$$

where we have put $R(t) = a(t)$ to get back to our original notation. We can also extract components such as $G^{\alpha\beta}$ shown in figure 4.2

and we can see that

$$G^{00} = 3\dot{R}^2(t)/R^2(t)$$

in units such that $c = G = 1$ (where G is Newton's gravitational constant).

Now we know that (in our cool, low pressure Universe)

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta}.$$

Substituting $H(t) = \dot{R}(t)/R(t)$ and noting that if the Robertson Walker metric is consistent with General Relativity and the stress-energy tensor we derived above then $T^{00} = \rho_{RW}$, and we see that

$$3H^2(t) = 8\pi\rho_{RW}$$

or

$$\rho_{RW} = 3H^2(t)/8\pi.$$

This then is the density of a spatially flat universe.

From Figure 4.2 we see that there are other potentially non zero components of the Einstein tensor. All non diagonal components are zero but

$$G^{11} = -\dot{R}^2(t) - 2R(t)\ddot{R}(t)$$

$$G^{22} = -r^2\dot{R}^2(t) - 2r^2R(t)\ddot{R}(t)$$

$$G^{33} = -r^2\sin^2\theta\dot{R}^2(t) - 2r^2\sin^2\theta R(t)\ddot{R}(t).$$

For these to be compatible with our stress-energy tensor then the following condition must hold

$$\dot{R}^2(t) + 2R(t)\ddot{R}(t) = 0.$$

```

<< componen.m
Loading Components for DOS/Windows . . .

=====
MathTensor (TM) 2.2 (DOS/Windows(R)) (June 1, 1994)
      Components Package
by Leonard Parker and Steven M. Christensen
Copyright (c) 1991-1994 MathSolutions, Inc.
Runs with Mathematica (R) Versions 2.X.
Licensed to one machine only, copying prohibited.
=====
Components["C:\Windows\Desktop\compinrw.m",
"C:\Windows\Desktop\compoutrw.m", "C:\Windows\Desktop\compoutw.out"]

The following tensors have been calculated and stored

in the file C:\Windows\Desktop\compoutrw.m in InputForm, and

in the file C:\Windows\Desktop\compoutw.out in OutputForm:

Metricg
MatrixMetricgLower
MatrixMetricgUpper
Detg
AffineG[ua,lb,lc]
RicciR[la,lb]
ScalarR

EinsteinG[la,lb]

RiemannR[la,lb,lc,ld]

WeylC[la,lb,lc,ld]

You can edit C:\Windows\Desktop\compoutw.out
to print a record of the results.

EinsteinG[-4, -4]

$$\frac{3 a'[t]^2}{a[t]^2}$$


EinsteinG[-1, -1]

$$\frac{-a'[t]^2 - 2 a[t] a''[t]}{c}$$


EinsteinG[-2, -2]

$$\frac{-r^2 a'[t]^2 - 2 r^2 a[t] a''[t]}{c}$$


EinsteinG[-3, -3]

$$\frac{-r^2 \sin[\theta]^2 a'[t]^2 - 2 r^2 a[t] \sin[\theta]^2 a''[t]}{c}$$


ScalarR

$$\frac{6 (a'[t]^2 + a[t] a''[t])}{c a[t]^2}$$


```

Figure 4.1 Calculating Einstein Tensor Components

```

EinsteinG[4, 4]

$$\frac{3 a'[t]^2}{c^2 a[t]^2}$$

EinsteinG[1, 1]

$$\frac{-a'[t]^2 - 2 a[t] a''[t]}{c a[t]^4}$$

EinsteinG[2, 2]

$$\frac{-a'[t]^2 - 2 a[t] a''[t]}{c r^2 a[t]^4}$$

EinsteinG[3, 3]

$$\frac{-\text{Csc}[\text{theta}]^2 a'[t]^2 - 2 a[t] \text{Csc}[\text{theta}]^2 a''[t]}{c r^2 a[t]^4}$$


```

Figure 4.2 Calculating Einstein Tensor Components

This equation governs the dynamics of the Universe and must hold in a low pressure universe, with zero cosmological constant, that is described by the spatially flat Robertson Walker metric.

We can again use Mathematica to plot the scale factor $R(t)$ against t . Figure 4.3 shows how this is done and the result.

We can see that the universe (with critical mass and thus spatially flat) will expand forever but at a decelerating rate. We cannot extend the graph in Fig 3 too far back in (cosmic) time as the low pressure, low temperature assumptions become invalid. However, assuming our Universe is spatially flat and has zero cosmological constant, then the above graph is likely to be a good representation of its dynamics apart from near the origin.

4.5 Conclusion

We have now shown that General Relativity admits the Robertson Walker metric as a solution to the field equations arising from the stress-energy tensor of low pressure, low temperature dust, provided the dynamics of the

```

(* This session is designed to plot the dynamics *)
(* of a low pressure critical density Robertson Walker *)
(* universe *)

solution =
  R[t] /. NDSolve[{(R'[t])^2 + 2*(R[t])*(R''[t]) == 0,
    R[1] == 1, R'[1] == .1}, R[t],
    {t, 1, 100000000}]
  {InterpolatingFunction[{{1., 1. x 10^8}}, <>][t]}
Plot[solution, {t, 1, 100000000}]

- Graphics -

```

Figure 4.3 The Dynamics of the Universe

universe (with $\Lambda = 0$, where Λ is the cosmological constant) are governed by the equation.

$$\dot{R}^2(t) + 2R(t)\ddot{R}(t) = 0.$$

Chapter 5

Geodesics of massive particles

5.1 Does the physics of cosmological redshift as applied to massive particles arise from the Robertson Walker metric?

We will now investigate massive particle geodesics (i.e. time-like as opposed to null) to demonstrate (or otherwise) that the physics of cosmological redshift as applied to massive particles arises from the (spatially flat) Robertson Walker description of the universe.

Geodesics are the world lines of particles in free fall (massive or massless). An individual geodesic is “selected” by the initial conditions of starting coordinates and velocity. All geodesics of massive particle obey the geodesic equation:

$$d^2x^\mu/d\tau^2 + \Gamma_{\nu\rho}^\mu(dx^\nu/d\tau)(dx^\rho/d\tau) = 0.$$

This follows quite simply from the definition of covariant differentiation and noting that for a particle in free fall the covariant derivative of the momentum 4-vector is zero.

For a particle in free fall the four-force vector $F^\mu = 0$ and thus

$$F^\mu = \frac{Dp^\mu}{D\tau} = 0$$

where (in the notation used by Kenyon [2]) $Dp^\mu/D\tau$ is the covariant derivative of the momentum 4-vector.

Now

$$\frac{Dp^\mu}{D\tau} = \frac{\partial p^\mu}{\partial \tau} + \Gamma_{\nu\rho}^\mu \frac{\partial p^\nu}{\partial \tau} p^\rho.$$

Substituting $p^\mu = m\partial x^\mu/\partial\tau$ gives the geodesic equation above.

In the course of calculating the components of the Einstein tensor, Mathematica has already generated all the metric connections we need to solve this (set of) equation(s).

The non zero connections are:

$$\Gamma_{tr}^r = \dot{a}(t)/a(t)$$

$$\Gamma_{\phi\phi}^r = -r \sin^2 \theta$$

$$\Gamma_{\theta\theta}^r = -r$$

$$\Gamma_{t\theta}^\theta = \dot{a}(t)/a(t)$$

$$\Gamma_{\phi\phi}^\theta = -\cos \theta \sin \theta$$

$$\Gamma_{\theta r}^\theta = -1/r$$

$$\Gamma_{t\phi}^\phi = \dot{a}(t)/a(t)$$

$$\Gamma_{\phi\theta}^\phi = \cot \theta$$

$$\Gamma_{\phi r}^\phi = 1/r$$

$$\Gamma_{\phi\phi}^t = r^2 a(t) (\sin^2 \theta) \dot{a}(t)$$

$$\Gamma_{\theta\theta}^t = r^2 a(t) \dot{a}(t)$$

$$\Gamma_{rr}^t = a(t) \dot{a}(t)$$

where, in the above, $a \equiv R$, $\dot{a} \equiv \dot{R}$.

If we consider a massive particle moving radially outward from a point on a trajectory such that $\theta = \phi = 0$, then we can be sure in an isotropic universe, such as is described by the Robertson-Walker metric, that at any

later time t , $\theta(t) = \phi(t) = 0$, for all t and thus $\dot{\theta}(t) = \dot{\phi}(t) = 0$ for all t (in fact any derivative of θ and ϕ with respect to any affine parameter, will be zero).

We can in all generality choose our comoving coordinate system such that the above condition holds (gauge fixing). In this case we simplify the task of solving the set of second order differential simultaneous equations and we are left with only two equations

$$d^2r/d\tau^2 + \Gamma_{tr}^r(dt/d\tau)(dr/d\tau) = 0$$

$$d^2t/d\tau^2 + \Gamma_{rr}^t(dr/d\tau)(dr/d\tau) = 0$$

or

$$d^2r/d\tau^2 + (\dot{a}(t)/a(t))(dt/d\tau)(dr/d\tau) = 0$$

and

$$d^2t/d\tau^2 + a(t)\dot{a}(t)(dr/d\tau)^2 = 0$$

and again $a \equiv R$, $\dot{a} \equiv \dot{R}$.

To solve these we need to know how a and thus \dot{a} vary with time. We have already solved this problem numerically with Mathematica. We can do the same analytically. We have already seen that in our critical mass universe

$$3H^2(t) = 8\pi\rho_{\text{critical}}$$

or

$$3\dot{R}^2(t) = 8\pi\rho_{\text{critical}}R^2(t).$$

Of course the value of ρ_{critical} itself varies with $H(t)$ (or $R(t)$). At some arbitrary time (for instance now) in a spatially flat universe, where $t = t_0$ and $R(t) = R(t_0)$ we can define $\rho_{\text{critical}0}$ as the critical density at $t = t_0$. The value of ρ_{critical} at other times will therefore be given by

$$\rho_{\text{critical}} = \rho_{\text{critical}0}(1/(R^3(t)/R^3(t_0))).$$

We can now write

$$3\dot{R}^2(t) = 8\pi\rho_{\text{critical}0}(1/(R^3(t)/R^3(t_0)))R^2(t)$$

or

$$3\dot{R}^2(t) = 8\pi\rho_{\text{critical}0}R^3(t_0)/R(t) = C/R(t).$$

The solution of which is

$$R(t) = At^{2/3}$$

and therefore

$$\dot{R}(t) = \frac{2}{3}At^{-1/3}.$$

We can now substitute in our expressions for $R(t)$ and $\dot{R}(t)$ into the geodesic equations and solve them using Mathematica. Thus

$$d^2r/d\tau^2 + \left(\frac{2}{3}At^{-1/3}/At^{2/3}\right)(dt/d\tau)(dr/d\tau) = 0$$

$$d^2t/d\tau^2 + At^{2/3}\frac{2}{3}At^{-1/3}(dr/d\tau)^2 = 0$$

or

$$d^2r/d\tau^2 + \left(\frac{2}{3t}\right)(dt/d\tau)(dr/d\tau) = 0$$

$$d^2t/d\tau^2 + \frac{2}{3}A^2t^{1/3}(dr/d\tau)^2 = 0$$

Mathematica can't give us an analytical solution to these equations but Figure 5.1 shows how a numerical solution is obtained and plots the result. In this exercise the value of A has been set to unity.

In Figure 5.1 the horizontal axis is the co-moving radial dimension r and the vertical is “cosmic time”. Initial values were chosen for the particle velocity $dr/d\tau = 0.5$ and $dt/d\tau = 1.16$ and g represents the proper time along the geodesic τ .

We can see that the particle eventually loses all “peculiar” motion and comes to rest with respect to the local matter field i.e. $dr/dt \rightarrow 0$ as $t \rightarrow \infty$. We have already shown that once a particle is at rest with respect to the average local matter field then it remains so (i.e. co-moving world line are geodesics in the RW metric). This point is discussed further in section 5.3, where we try to answer the question “where does the kinetic energy go?”


```

NDSolve[{r'[g] + (2/(3*t[g])) t'[g] r'[g] == 0,
  t'[g] + (2/3) (t[g]^(1/3)) (r'[g])^2 == 0,
  t'[0] == 1.16, t[0] == 10000000000000, r'[0] == .5, r[0] == 0},
  {r, t}, {g, -11, 200000}]

```

```

NDSolve::ndsz : At g == 66140.1121642878742`, step size is
effectively zero; singularity suspected.

```

```

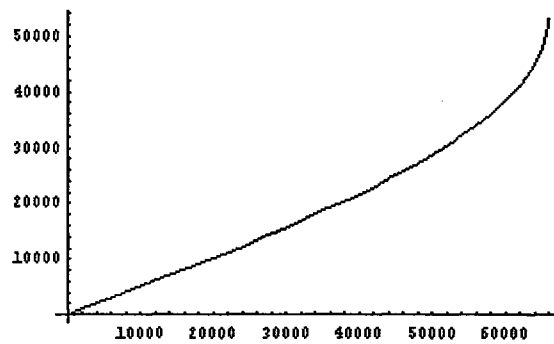
{{r→InterpolatingFunction[{{-11., 66140.1}}, <>],
  t→InterpolatingFunction[{{-11., 66140.1}}, <>]}}

```

```

Plot[Evaluate[r[g] /. %55], {g, -11, 66000}]

```



- Graphics -

Figure 5.1 Geodesics for Massive Particles

5.2 Choice of initial conditions

It is worth noting the detail of the choice of the initial conditions in the Mathematica session in the last section. Choices for $t[0]$, the proper time of the particle at cosmic time $\tau = 0$ and $r[0]$, the radial coordinate also at $\tau = 0$ can be chosen for convenience. However the values of the first derivatives must be mutually consistent with Special Relativity.

In the example we set

$$r'[0] = 0.5 = dr/d\tau_{\tau=0}$$

(half the speed of light as observed in the local co-moving frame). In this case the proper time of the particle will be given by the special relativistic “time dilation” formula, thus

$$dt = d\tau / \sqrt{1 - \frac{v^2}{c^2}}$$

where $v = dr/d\tau$ and $c = 1$, at cosmic time $\tau = 0$, $v[0] = dr/d\tau_{\tau=0} = 0.5$, so

$$t'[0] = dt/d\tau_{\tau=0} = 1/\sqrt{1 - 0.25} = 1/\sqrt{.75} = 1.155 \simeq 1.16.$$

5.3 Cosmological redshift as it applies to massive particles.

We have demonstrated that massive particles with non zero “peculiar” velocity with respect to a local co-moving reference frame will move through the universe (of otherwise uniform co-moving dust) gradually losing peculiar motion with respect to the local co-moving frame, until at some locality they essentially become part of the co-moving matter field themselves. There are several important observations to be made about this effect.

1. This effect is of course “frame dragging” on a large scale. Seen from the point of view of the particle, the average local matter is rushing past in one direction. The particle will gradually accelerate in the same direction (with respect to the frame in which we first viewed the particle), gaining energy from the local matter field until it is co-moving with it.

2. From the point of view of a co-moving observer, the situation is also interesting. Initially the particle will be seen to decelerate with respect to the local matter field (It does its own small bit of “frame dragging”, donating its kinetic energy to the matter field. See paragraph 3 below). As it moves further from the observer, the local matter field is also moving away from the observer and the velocity of the particle will be seen to approach the co-moving velocity of its local matter field.

3. The exchange of energy between the “peculiar motion” of a particle and the local surrounding matter, described in 1 and 2 above provides a mechanism for the kinetic energy of particles with randomly oriented high velocities in an early hot universe, to be converted to a uniform expansion in a later, cooler one.

4. Without this mechanism, we would expect to see particles of matter with relativistic velocities with respect to the local average. In fact we do see charged particles (cosmic rays) arriving at the earth with relativistic velocities but not larger neutral particles or “lumps” of matter. See Chapter 11 section 3 for a discussion of possible further work in this area.

5.4 How is all this affected by having a non-zero cosmological constant?

The latest evidence (see chapter 6) is for a non zero value for the cosmological constant, so we now “revisit” the derivation of the equations for the critical

mass and the dynamics of the spatially flat universe but this time with the inclusion of a non-zero cosmological constant.

Following a very similar treatment to that used in previous sections, Kenyon ([2] Ch 11, Sect 3) arrives at the following equations for Robertson Walker space-time with positive ($k = 1$), zero ($k = 0$) and negative ($k = -1$) curvature. In these equations, Λ is the cosmological constant originally introduced by Einstein into the field equations to support the idea of a static universe:

$$3\dot{R}^2/R^2 + 3kc^2/R^2 - c^2\Lambda = 8\pi G\rho$$

and

$$-2\ddot{R}/R - \dot{R}^2/R^2 - kc^2/R^2 + c^2\Lambda = 8\pi Gp/c^2$$

where p is the pressure which we have previously taken as zero. In a spatially flat universe (the reality of which is strongly supported by recent observational evidence and analysis), these simplify to

$$3\dot{R}^2/R^2 - c^2\Lambda = 8\pi G\rho$$

(compare with $3\dot{R}^2/R^2 = 8\pi G\rho$ obtained earlier) and

$$-2\ddot{R}/R - \dot{R}^2/R^2 + c^2\Lambda = 8\pi Gp/c^2$$

Thus the dynamical equation for a low pressure flat RW universe is

$$-2\ddot{R}/R - \dot{R}^2/R^2 + c^2\Lambda = 0$$

or

$$\dot{R}^2(t) + 2R(t)\ddot{R}(t) - c^2\Lambda R^2 = 0$$

(compare with $\dot{R}^2(t) + 2R(t)\ddot{R}(t) = 0$ obtained earlier).

Whereas previously we had (with ρ_0 representing the critical matter density for a “flat” universe with zero cosmological constant) $\rho_0 = 3H^2(t)/8\pi G$, we now have

$$\rho = 3H^2(t)/8\pi G - \Lambda/8\pi G \tag{5.1}$$

in units where $c = 1$. Thus in order to have a spatially flat universe with a non-zero cosmological constant, the matter density is reduced by an amount $\Lambda/8\pi G$.

Rearranging equation 5.1 gives $\rho + \Lambda/8\pi G = 3H^2(t)/8\pi G$ and dividing by ρ_0 gives

$$\rho/\rho_0 + \Lambda/8\pi G\rho_0 = 3H^2(t)/8\pi G/\rho_0 = 1$$

which we can write as

$$\Omega_M + \Omega_\Lambda = 1$$

where $\Omega_M = \rho/\rho_0$ and $\Omega_\Lambda = \Lambda/8\pi G\rho_0$.

This then is the condition for a spatially flat universe. It is to the observational evidence regarding the actual values of Ω_M , Ω_Λ , and crucially $\Omega_M + \Omega_\Lambda$ that we now turn.

Chapter 6

Justification for the concentration on the spatially flat ($k = 0$) Robertson Walker model of the universe.

6.1 COBE - an introduction

In this section the evidence for the Universe being (at least approximately) spatially flat is examined.

In November 1989, NASA launched a satellite called COBE. It carried sensitive detectors to study the full spectrum of the cosmic microwave background radiation. Initially, COBE's task was to confirm that the background radiation was isotropic and that its spectrum was "black body". After taking known local motions of the earth around the sun and the sun around the galactic centre into account, these aims were achieved to a very high degree of confidence. The instrument on board the COBE spacecraft was known as the "Far Infrared Absolute Spectrophotometer" or FIRAS. The final results, corrected for the local movement of the earth are shown in the graph, Fig 6.1. [5]

The solid curve shows the expected intensity from a single temperature black body spectrum. The FIRAS data were taken at 34 positions equally spaced along the curve. The FIRAS data match the curve so exactly that it

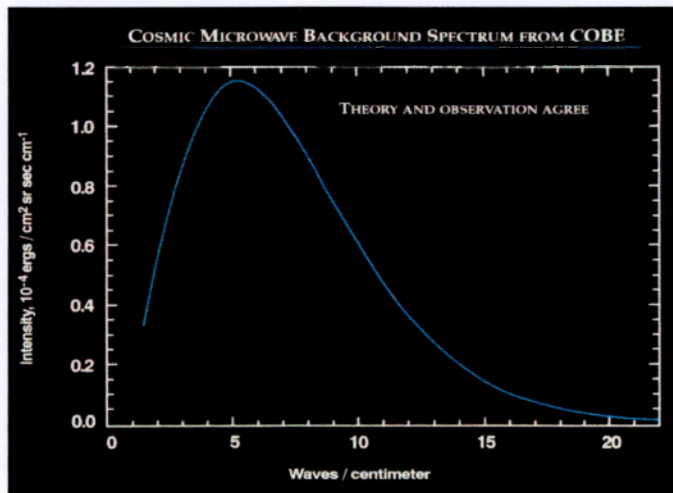


Figure 6.1 Cosmic Microwave Background Radiation Black Body Spectrum

is impossible to distinguish the data from the theoretical curve when plotted at this scale (and this printing resolution).

The next phase was to look for small variations in the otherwise isotropic black body temperature, the reason being that detailed study of the angular scale of the anisotropies in the background radiation might reveal the value of $\Omega_M + \Omega_\Lambda$ and thereby the geometry of the Universe on a grand scale.

6.2 Anisotropy and geometry

The Universe is not populated by a uniform homogeneous dust cloud! Structure exists in the Universe over a wide range of scales, from individual elementary particles and atoms to galaxies and superclusters (of galaxies). In the very early Universe when the temperature was very high, the matter would have consisted of disassociated charged particles. Such a plasma would have been opaque to electromagnetic radiation and the radiation and matter would have been in thermal equilibrium. As the Universe expanded and cooled the free electrons would be captured by charged nuclei and the

Universe would have become largely transparent to radiation. This era in the development of the Universe is known as the era of recombination and the radiation we receive now from the Cosmic Microwave Background Radiation (CMBR) is the radiation that existed then. Its black body spectrum is a result of (and evidence for) the state of thermal equilibrium at the time.

Since the existence of structure in the Universe is thought to have been “seeded” by small density/temperature fluctuations in the otherwise isotropic Universe, the smallness of the observed anisotropies in the CMBR temperature is very good evidence for both the homogeneity of the Universe at the era of recombination and for the reasonableness of the assumption of homogeneity on a very large scale at later stages (an assumption we have used throughout this analysis). Note - we have already shown in chapter 3.1, that isotropy as seen by all observers through out the Universe, implies a homogeneous Universe.

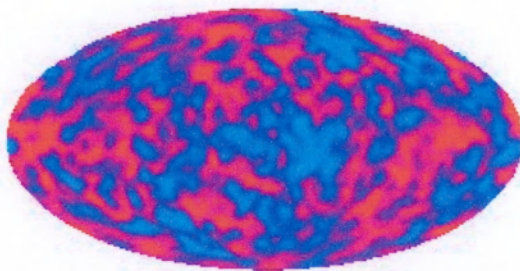


Figure 6.2 COBE CMBR Temperature Fluctuation Map

The COBE CMBR temperature variation map, Fig 6.2 [11], shows temperature fluctuations after the dipole pattern due to the motion of the Earth and radiation from the Milky Way have been subtracted. The RMS value of the anisotropy is $30\mu\text{K}$. This amounts to an amplitude of 1 part in 10^5 .

The uniformity of the CMBR leads one to suppose that all parts of the Universe were in thermal equilibrium in the early stages of its evolution. This requires that all parts of the Universe must have been at some point in “causal contact”, which given a finite speed of light c , means that the Universe was extremely compact. It was proposed, initially by Guth [6], that the

early Universe underwent a short but extremely rapid period of exponential expansion known as Inflation. Throughout this phase, small quantum fluctuations are expected to have occurred giving rise to density/temperature fluctuations at a range of scales [7] and [8].

As Inflation drove areas of the Universe out of causal contact with each other, temperature differences could no longer be corrected and would therefore persist as the Universe expanded further. The largest such areas would, of course, have begun earliest in the evolution of the Universe and would have given rise to anisotropies in the CMBR that were c times the age of the Universe at recombination across. This equates to 300,000 light years [9], Page 170. The cosmological redshift tells us the factor by which the Universe has expanded since the CMBR left the “last scattering surface” and one can therefore work out the angular size of such an anisotropy as seen by a microwave detector. In a spatially flat universe, the angular size would be a little less than 1 degree. If it has positive curvature, the angular size will be less than that and if it is negative, the angular size will be greater than one degree [9]. The angular size of the largest anisotropy therefore gives us information about the value of $\Omega_M + \Omega_\Lambda$.

We have noted above the relationship between density and temperature fluctuations. The following diagram (Fig 6.3) and text are taken from Ed Wright’s pages of the UCLA web site [11] and provide a very good description of this relationship as being due to “dips” in the (cosmic) gravitational potential.

“These dense regions should affect the temperature of the microwave background. Sachs and Wolfe [10] derived the effect of the gravitational potential perturbations on the CMB. The gravitational potential, $\phi = -GM/r$, will be negative in dense lumps, and positive in less dense regions. Photons lose energy when they climb out of the gravitational potential wells of the lumps:

This conformal space-time diagram above (see Fig 6.3) shows lumps as gray vertical bars, the epoch before recombination as the hatched region, and the gravitational potential as the color-coded curve $\phi(x)$. Where our past

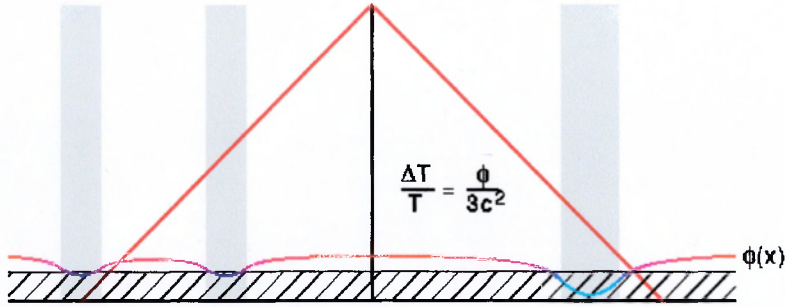


Figure 6.3 CMBR Temperature Fluctuations and Gravitational Potential

*lightcone intersects the surface of recombination, we see a temperature perturbed by $dT/T = \Phi/(3 * c^2)$. Sachs and Wolfe predicted temperature fluctuations dT/T as large as 1 percent, but we know now that the Universe is far more homogeneous than Sachs and Wolfe thought. So observers worked for years to get enough sensitivity to see the temperature differences around the sky.” [11]*

6.3 Multipole analysis

In this section we look at how the data is gathered and analyzed. The usual way to analyze the observations is in terms of “multipoles”. Dipole anisotropy looks for differences between “one half of the sky and the other”. The division between the two halves is rotated around to find the maximum anisotropy and this then yields the “dipole contribution”. This was first measured by Conklin in 1969 and was reported to the IAU Symposium no.44 held in Sweden in 1970 [12].

The picture (Fig 6.4 [11]) shows the dipole anisotropy as detected by COBE and is a better measurement than Conklin’s 1969 measurement.

Further divisions of the sky yield higher multipole contributions. The pole index usually denoted by the letter l , with $l = 0$ being the monopole case and $l = 1$, the dipole case. In general, these multipoles are referred to

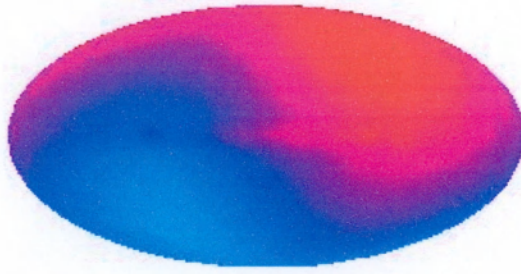


Figure 6.4 COBE CMBR Dipole Anisotropy

as “ l -poles”. In the dipole case $l = 1$, the angular resolution is 180 degrees. To obtain the approximate angular resolution at other values of l (excluding $l = 0$) we use

$$\text{Resolution} = 180/l$$

giving the answer in degrees.

For $\Omega_M + \Omega_\Lambda = 1$ we expect a peak in the RMS anisotropy value at $l \simeq 200$ [9] page 174. This gives us a typical angular size of $180/200 = 0.9$ degrees.

COBE could not observe angular scales finer than about 10 degrees ($l = 18$) and perhaps the best that can be said about the data is that it does not contradict the idea that the Universe is spatially flat.

6.4 Further studies of CMBR anisotropy

Two ground based experiments have recently reported results. These are:

- 1 The Very Small Array.
- 2 Cosmic Background Imager.

The Very Small Array (VSA) is a synthesis telescope designed to image faint structures in the cosmic microwave background on degree and sub-

degree angular scales. It is situated at the Teide Observatory, Tenerife at an altitude of 2400m. A detailed description is given in R A Watson et al [13].

In May 2002, José Alberto Rubiño-Martin et al [14] reported on Cosmological Parameter Estimation based primarily on the VSA results. In this paper, they come to the conclusion that, based on VSA and COBE data alone,

$$\Omega_{\text{TOT}} = \Omega_{\text{M}} + \Omega_{\Lambda} = 1.03^{+0.12}_{-0.12}.$$

The same group then add in “type Ia supernovae constraints” , which we will discuss in Chapter 6, Section5 and arrive at

$$\Omega_{\text{M}} = 0.32^{+0.09}_{-0.06}$$

and

$$\Omega_{\Lambda} = 0.71^{+0.07}_{-0.07}$$

and finally combining all the recent CMB experiments with evidence for H_0 from the HST (Hubble Space telescope) they obtain, independently of the supernova data

$$\Omega_{\text{M}} = 0.28^{+0.14}_{-0.07}$$

and

$$\Omega_{\Lambda} = 0.72^{+0.07}_{-0.13}.$$

The Cosmic Background Imager is a special purpose radio telescope designed to study the cosmic microwave background. It is located in the Chilean Andes at an altitude of 5080m. Details of the design of the Imager are found in the paper by Padin et al [15].

This telescope has been designed to examine the detail in the CMBR up to $l = 3500$. Sievers et al (May 2002) [16], using CBI data in combination with COBE data, claim

$$\Omega_{\text{TOT}} = 0.99^{+0.12}_{-0.12}(1\sigma)$$

and

$$\Omega_{\Lambda} = 0.64^{+0.11}_{-0.14}.$$

Additionally, they claim excellent agreement with the BOOMERANG, DASI and MAXIMA results in the region $l < 1000$.

MAXIMA and BOOMERANG are balloon borne bolometric instruments designed to measure the CMBR anisotropy on angular scales to less than a degree. These projects are part of the NSF Centre for Particle Astrophysics.

Balbi et al [17] report on results from MAXIMA alone giving

$$\Omega_{\text{TOT}} = 1.0^{+0.15}_{-0.30}$$

and

$$0.45 < \Omega_{\Lambda} < 0.75.$$

Jaffe et al (April 2001) [18], taking MAXIMA-1, BOOMERANG & COBE/DMR measurements together arrive at

$$\Omega_{\text{TOT}} \simeq 1.11 \pm 0.07^{(+0.13)}_{(-0.12)}.$$

Descriptions of the MAXIMA balloon borne experiments can be found on <http://aether.lbl.gov/www/projects/max/> .

A nice summary of the fit of the data to the expected curve is given by E. Wright [www.astro.ucla.edu/~wright] [11] and is shown in Fig 6.5.

More recently (Jan 2003), Benoit et al have reported in two papers (The Cosmic Microwave Background Anisotropy Power Spectrum measured by ARCHEOPS [19] and Cosmological Constraints from Archeops [20]) their results and conclusions from the ARCHEOPS experiment. ARCHEOPS is a balloon borne instrument consisting of a 1.5m aperture diameter telescope with an array of 21 photometers maintained at about 100mK that operate in four frequency bands at 143, 217, 353 and 545 GHz. The data were taken during the Arctic night of February 7th, 2002. The multipole range covered was $l = 15 \rightarrow 350$.

In the second paper listed above, Benoit et al report

$$l_{\text{PEAK}} = 220 \pm 6$$

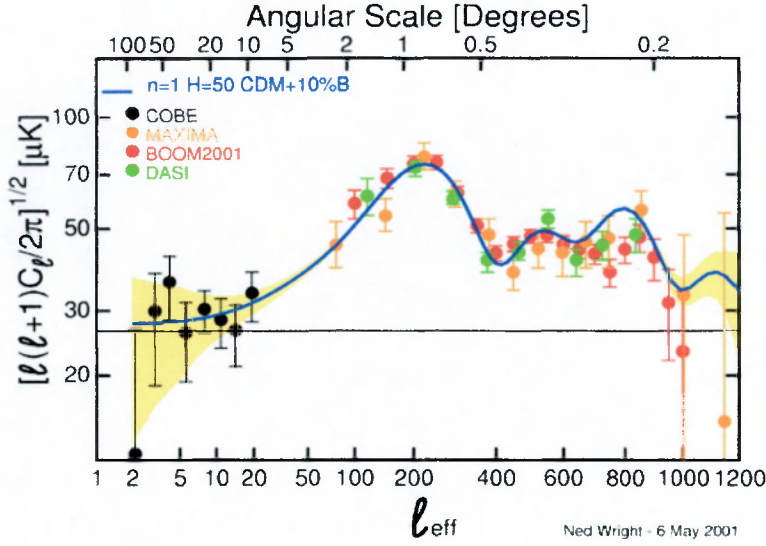


Figure 6.5 Temperature Fluctuations and Angular Resolution

with an amplitude,

$$\delta T = 71.5 \pm 2.0 \mu K.$$

From ARCHEOPS data with the HST constraint for $H_0 = 72 \pm 8 \text{ km/s/Mpc}$ (See Freedman et al. 2001) [21] they conclude that

$$\Omega_{\text{TOT}} \simeq 0.96^{+0.09}_{-0.04}.$$

The paper give a range of values by combining ARCHEOPS data with data from other sources. All such combinations of the data support the notion of an essentially flat Universe.

6.5 What about Ω_M and Ω_Λ separately?

Donald Goldsmith's book, "The Runaway Universe" [9], provides a good summary of the evidence for a non-zero cosmological constant. In the last section, we gave values for $\Omega_{\text{TOT}} (= \Omega_M + \Omega_\Lambda)$ but in some cases also gave separate values for Ω_M and Ω_Λ . We will look briefly at how these are arrived at.

In 1993, Mark Phillips, at the Cerro Tololo Inter-American Observatory in Chile, showed in a paper that type Ia supernovae (SN Ia) could be used as highly luminous “standard candles”.

He used a series of other astronomer’s observations of relatively nearby SN Ia’s that had appeared in galaxies with reasonably well known distances from the Milky Way and showed that there was a good correlation between peak luminosity and the rate at which luminosity declined, i.e. the most luminous SN Ia’s fade least quickly.

Type Ia supernovae are thought to be white dwarf stars that accumulate matter from companion stars in binary systems until they reach the Chandrasekhar limit (of about 1.4 solar masses). Then a massive fusion reaction throughout the (degenerate) matter of the white dwarf occurs simultaneously, producing an explosion of known energy output, as the starting mass is the same in all cases.

Further work by Riess refined the use of SN Ia’s as standard candles and this technique was used by the “Supernova Cosmology Project” headed by Dr Saul Perlmutter and the “High-Z Supernova Search Team” initiated by Robert Kirshner in collaboration with Brian Schmidt.

These results of both teams are in good agreement and strongly favour a non zero +ve Ω_Λ . Fig 6.6 [11] shows results from the Supernova Cosmology Project.

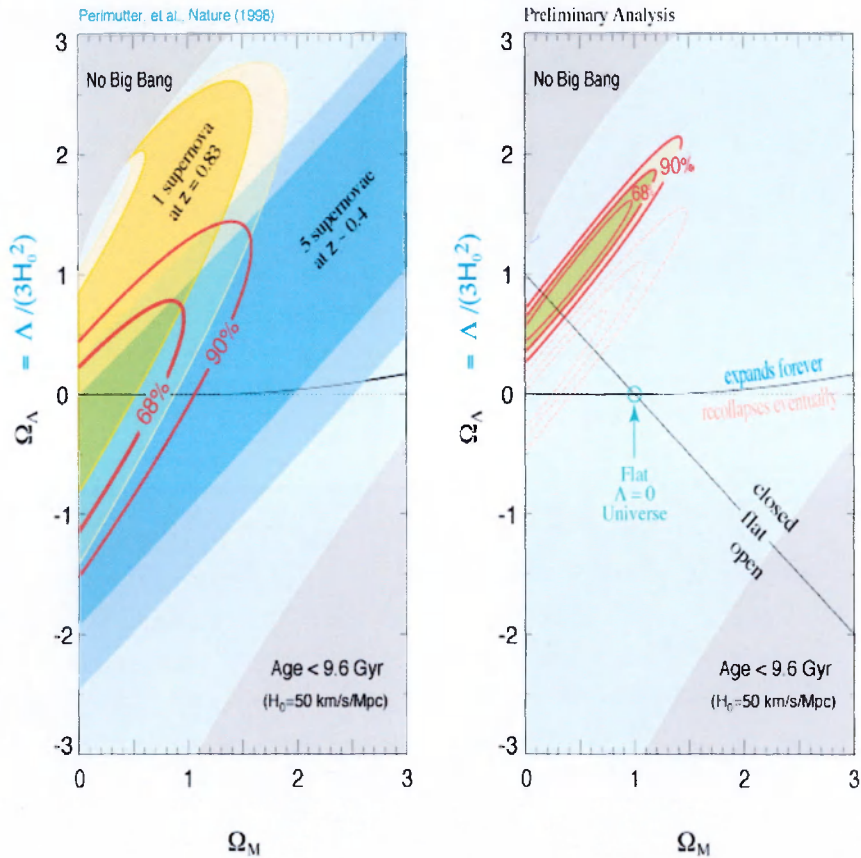
The accumulation of evidence overall seems to point to a spatially flat universe with $\Omega_M \simeq 0.3$ and $\Omega_\Lambda \simeq 0.7$.

6.6 The WMAP mission

The Wilkinson Microwave Anisotropy Probe is a spacecraft that has two telescopes focusing microwave radiation onto ten microwave receivers, five per telescope, arranged to pick up signals in five bands from 22GHz to 90GHz.

Results: Ω vs Λ from 6 supernovae

Results: Ω vs Λ from 40 supernovae



These two plots show the best-fit confidence regions on the Ω_M -vs- Ω_Λ plane for the 6-supernova fit presented in the Nature (1998) paper and for a more extensive 40-supernova fit (preliminary analysis). The left plot demonstrates that with a range of redshifts from 0.4 to 0.85, the approximately straight slope of the confidence region at a given redshift begins to rotate, allowing an intersection region (shown in green) to isolate measurements of Ω_M and Ω_Λ separately, not just in linear combination (see Goobar & Perlmutter, Ap.J. 1995). With the larger sample of supernovae shown on the right plot, the statistical uncertainty is now small enough—and the confidence regions narrow enough—that the systematic uncertainty is the dominant source of error. The dashed-line confidence region on the right plot shows our preliminary estimate of this systematic uncertainty (shown in the direction of 0.2 lower apparent magnitudes for the high redshift supernovae). Further analysis should reduce this uncertainty. The best-fit confidence region (in green on the right plot) is centered at $\Omega_M = 0.5$, $\Omega_\Lambda = 1.0$. This confidence region lies along the line of $\Omega_\Lambda = \Omega_M + 0.5$, which is not parallel to the lines of constant deceleration $q_0 = \Omega_M/2 - \Omega_\Lambda$. Note that the confidence regions do not include the "standard model" inflationary universe with no cosmological constant (shown as a green circle at the intersection of the flat-universe line and the $\Lambda = 0$ line). The confidence regions do suggest that we live in a universe that will expand forever.

Figure 6.6 Ω vs Λ Results from the Supernova Cosmology Project

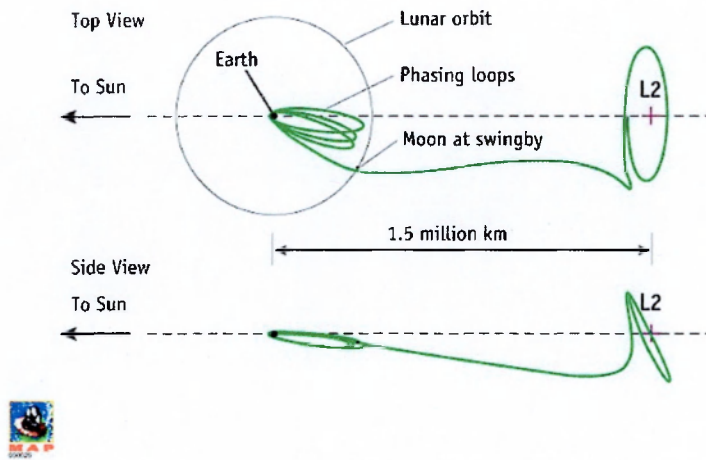


Figure 6.7 WMAP orbit at L2 Sun-Earth Lagrange Point.

The use of five bands helps eliminate signals from our own Galaxy. The telescopes are angled 140° apart and make accurate differential temperature measurements. The angular resolution achieved is 0.3° . The spacecraft is positioned at the L2 Sun-Earth Lagrange point. This is where the force on any object due to the Sun and the Earth/Moon system provides just the right centripetal force for the object to revolve around the Sun with the Earth, staying a constant $1.5 \times 10^9 \text{m}$ from the Earth. In fact the spacecraft follows a relatively close orbit about the L2 point. Figure 6.7 taken from the NASA website <http://map.gsfc.nasa.gov> illustrates the orbit. [43]

The use of the Lagrange point means that the whole sky can be mapped while the telescopes remain pointed away from the Earth and the Sun. Approximately 30% of the sky is mapped per day and the full sky is covered every six months.

Two recent papers publish estimates of cosmological parameters based on the WMAP data:

Bennett, C. L. et al. First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Preliminary Maps and Results. [44]

Spergel, D. N. et al. First Year Wilkinson Microwave Anisotropy Probe

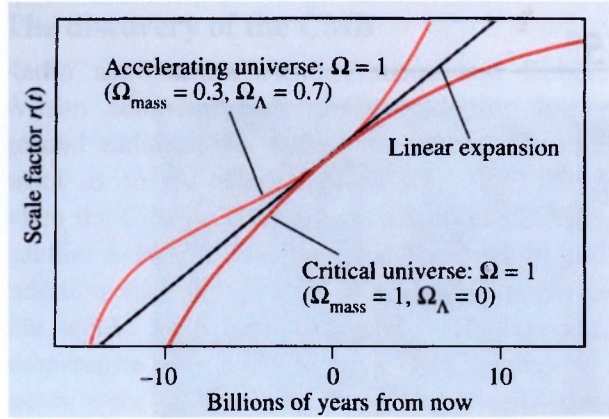


Figure 6.8 Dynamics of the Universe with $\Omega_{\text{TOT}} = 1$ and $\Omega_{\Lambda} = 0.7$ compared with $\Omega_{\Lambda} = 0$.

(WMAP) Observations: Determination of Cosmological Parameters. [45]

Both of the above have (as of 14th May 2003) been submitted to the Astrophysical Journal. These results are the most accurate to date and the relevant numbers as far as this project goes are:

$$\Omega_{\text{TOT}} = 1.02^{+0.02}_{-0.02}$$

$$\Omega_{\Lambda} = 0.73^{+0.04}_{-0.04}.$$

A very recent (May 2003), short review, by Ian Morison (of the Jodrell Bank Observatory), of the data and the resulting dynamics of the universe, based on MAP, Very small Array, MAXIMA, Boomerang and other sources, was published in Physics Education. Figure 6.8 shows the “latest view” of the dynamics of the Universe [46].

Chapter 7

Review of text book descriptions of cosmological redshift.

7.1 Introduction.

Cosmological redshift is an accepted fact of the Universe and whilst many textbooks on Cosmology, General Relativity and related subjects describe the effect to a greater or lesser extent, apart from this investigation, it is not (to the knowledge of the author) the subject of any current research. Kenyon [2] page 151 is typical of many in merely stating a relationship between scale factor R and wavelength as follows.

“Measurements of the Hubble constant give information on the behaviour of R . The wavelengths appearing in the redshift formulae depend on R ; as R increases so does the wavelength λ . Thus

$$\begin{aligned} z &= \frac{\lambda - \lambda_e}{\lambda_e} \\ &= (R - R_e)/R_e \\ &= \frac{\dot{R}\Delta t}{R} \end{aligned}$$

where Δt is the travel time between the source galaxy and Earth.”

A similar treatment is given in:

- 1 The cosmological Background Radiation - M. Lachièze-Rey and E. Gunzig, [22] page 12.
- 2 Modern Astrophysics - B. W. Carroll and D. A. Ostley [23] page 1171.
- 3 Introduction to Cosmology - Matts Roos [24] page 21.
- 4 Cosmological Inflation and Large-Scale Structure - A. R. Liddle and D. H. Lyth [25] pages 14,15.
- 5 A first course in General Relativity - B. F. Schutz [3] page 330.

Other texts vary in their treatment of the subject and a selection are briefly reviewed below.

7.2 Cosmological redshift in other textbooks.

J. L. Martin's "General Relativity" [26] Section 11.4 states a relationship between period and scale factor using the following argument, based on the Robertson Walker metric. In the passage quoted below, the term 'fundamental observer' means an observer co-moving with respect to the local matter field.

"Imagine that a fundamental observer A at the centre of coordinates $x = 0$ transmits, at epoch t_A a pulse of light which is received by a second fundamental observer B at position x_B and epoch t_B . On account of the spherical symmetry, along the null world line of the pulse both θ and ϕ are constant; consequently - whatever the choice of k :

$$0 = dt^2 - R^2 dx^2$$

which gives on integration

$$\int_{t_A}^{t_B} dt/R(t) = x_B.$$

If A emits a further pulse at a slightly later epoch $t_A + \Delta t_A$ and B receives it at $t_B + \Delta t_B$, then (since x_B does not change)

$$\Delta \int_{t_A}^{t_B} dt/R(t) = \frac{\Delta t_B}{R(t_B)} - \frac{\Delta t_A}{R(t_A)} = 0.$$

If we think of successive ‘pulses’ as the wave-crests of emitted radiation, we may take Δt_A to be the period of a photon transmitted by A and Δt_B to be the period of the same photon as perceived by B. If the Universe is *expanding*, that is, if

$$R(t_B) > R(t_A)$$

then

$$\Delta t_B > \Delta t_A$$

that is the *period of the photon increases*. This is the **cosmological redshift**.”

This explanation (in the integration $\Delta \int_{t_A}^{t_B} dt/R(t)$ along the null world line of a photon) treats cosmological redshift as a process that occurs during propagation of a photon. It is this view of cosmological redshift that this investigation challenges.

Similar explanations are given in:

- 1 Principles of Cosmology and Gravitation - M. V. Berry [27] page 107-108.
- 2 Gravitation and Cosmology - Steven Weinberg [28] page 415-416.
- 3 Modern Astrophysics - B. W. Carroll and D. A. Ostley [23] page 1265-1266.
- 4 Introduction to Cosmology - J. V. Narlikar [29] pages 92-94
- 5 Cosmology and Particle Astrophysics - L. Bergström and A. Gobar [30] page 72 ex 4.3.2.
- 6 Introducing Einstein’s Relativity - R. D’Inverno [31] page 325-325

7.3 Further “physical” explanations of cosmological redshift

“Physical” explanations of cosmological redshift are given in other text books. E. Harrison in his excellent book “Cosmology” [32], uses, in Chapter 17, the notion of a “cosmic box” to demonstrate cosmological or expansion redshift. In “Cosmological Physics”, J. A. Peacock explains redshift mathematically in a way similar to that described in the previous section but also uses as a model the “cosmic box” approach.

On page 72, Peacock [1] says *“Suppose we trap some radiation inside a box with silvered sides that expands with the universe. At least for an isotropic radiation field, the photons trapped inside are statistically equivalent to any that would pass into this space from outside. Since the walls expand at $v \ll c$ for a small box, it is easily shown that the Doppler shift maintains an adiabatic invariant which is the ratio of wavelength to box side and so radiation wavelengths increase as the universe expands. This argument also applies to quantum-mechanical standing waves: their momentum declines as $a(t)^{-1}$.”*

Of course a quantum mechanical standing wave in a box might be the wavefunction $\Psi(x)$ of a photon and may be considered as the sum of forward and backward travelling waves $\Psi^+(x, t)$ and $\Psi^-(x, t)$ and thus represents the case of a photon traversing the box with repeated reflections from the walls. We know (V. B. Braginsky and F. Khalili Quantum Measurement [33] page 11), that the photon mean frequency will decrease on reflection by an amount

$$\delta\omega/\omega = 2v_x/c$$

if the wall is moving in the x direction with velocity v_x . This leads to the momentum / wall side size relationship stated by Peacock (i.e. “their momentum declines as $a(t)^{-1}$ ”).

It is clear then that if photons have mean energy $\hbar\omega$ before reflection then they will have a mean energy of $\hbar\omega(1 - 2v_x/c)$ after reflection [33] Fig 1.5.

This explanation of cosmological redshift implies a repeated loss of energy from the photon field as the photons travel through space.

The case against cosmological redshift being a process of gradual (though quantized) loss of energy by photons was put succinctly by Zel'dovich. The argument is reported in "Gravitation (Misner Thorne and Wheeler)" [34] page 775. In it he says *"Let us suppose that a photon decays $\gamma \rightarrow \gamma' + k$ giving up a small part of its energy to some particle k . It follows from the conservation laws that k must move in the direction of the photon (this, by the way, avoids smearing out) and must have zero rest mass. Because of the statistical nature of the process, however, some photons would lose more energy than others and there would be a spectral broadening of the lines, which is also not observed."*

By "broadening" Zel'dovich means that spectral absorption lines from distant sources would become less distinct or "sharp". This is not the case. If it were, then spectroscopy would be applicable only to the nearest sources. In addition, if k , above did not necessarily travel in the same direction as the photon then by similar arguments, all images of distant sources would be blurred ("smearing out"). Luckily, this is not what is observed.

Any interaction of a photon with any other field, in quantum theory as it is now understood, must involve the transfer of energy and momentum and can be thought of as being described by the relation $\gamma \rightarrow \gamma' + k$ and is therefore subject to the argument put forward by Zel'dovich. To the author, it seems that the inescapable conclusion is that cosmological redshift is best understood as the result of "time dilation". Support for this view come from a paper by Leibundgut et al [47]. The abstract from this paper reads as follows. *"The light curve of a distant Type Ia supernova acts like a clock that can be used to test the expansion of the universe. SN 1995K, at a spectroscopic redshift of $z = 0.479$, provides one of the first meaningful data sets for this test. We find that all aspects of SN 1995K resemble local Type Ia supernova events when the light curve is dilated by $(1+z)$, as prescribed by cosmological expansion. In a static, nonexpanding universe, SN 1995K would represent*

a unique object with a spectrum identifying it as a regular Type Ia supernova but with a light-curve shape and luminosity that do not follow the well established relations for local events. We conclude that SN 1995K provides strong evidence for an interpretation of cosmological redshifts as being due to universal expansion. Theories in which photons dissipate their energy during travel are excluded, as are age-redshift dependencies."

Chapter 8

Gravitational potential, redshift and metric components in a spatially flat Robertson Walker universe

8.1 Introduction

This section is an attempt to understand cosmological redshift from the point of view of it being caused by global changes in gravitational potential. In doing so it asserts that time dilation between co-moving frames at the times of emission and reception is the cause of the perceived frequency shifts, this time dilation being the result of a potential difference between the frames at the time of emission and detection. This mechanism is analogous to that investigated by Pound and Rebka in their “classic” experiment. It avoids the “pitfall” of having the radiation undergo some “process” of redshift during propagation.

8.2 “Textbook” energy balance analysis

For a typical analysis we can follow Kenyon [2]. We consider a co-moving spherical surface of co-moving radial coordinate r and therefore of proper

radius $rR(t)$. In a spatially flat universe (or one of small curvature) and if we chose r to be small enough to keep the radial velocity small, then we can use Newtonian mechanics to analyze the situation.

We consider the mass contained within a small (co-moving) volume dV at the surface of the sphere. The mass contained within this volume is $\rho dV = m$. The kinetic energy associated with this mass, as seen from the co-moving frame at rest with respect to the centre of the sphere is $mr^2\dot{R}^2(t)/2$. The total mass contained within the sphere is $(4\pi r^3 R^3(t)\rho)/3$ and the gravitational potential energy of a mass m at a distance $rR(t)$ from a central mass $(4\pi r^3 R^3(t)\rho)/3$ is $-mG(4\pi r^2 R^2(t)\rho)/3$. (The potential is $-GM/rR$ and $M = (4\pi r^3 R^3(t)\rho)/3$).

The total energy of the mass m in the volume dV is therefore

$$mr^2\dot{R}^2(t)/2 - mG(4\pi r^2 R^2(t)\rho)/3.$$

Setting this quantity to zero gives us the equation

$$r^2\dot{R}^2(t)/2 - G(4\pi r^2 R^2(t)\rho)/3 = 0$$

or

$$3\dot{R}^2(t) - 8\pi G\rho R^2(t) = 0$$

so

$$\rho = 3H^2(t)/8\pi G$$

as $H = \dot{R}/R$.

This, as we saw in a previous section, is the critical mass density that is required for a universe with the spatially flat RW metric.

Thus we see that the spatially flat RW metric universe is one in which the total energy of the matter in a sphere about any point is zero.

8.3 The problem of defining a local measure of the “kinetic energy” of expansion

It seems reasonable to believe that the gravitational potential at any point (all points) is increasing with time and thus the gravitational potential energy in a local (co-moving) volume is increasing. It would be satisfying to be able to equate this to a decreasing “kinetic energy term”, for the matter in a local volume. However, in the RW metric, local matter (on average) has no velocity. In using a model in which the pressure is zero, we have assumed that the present day expansion of the Universe is not caused by radiation pressure. We might look at the idea that the expansion is due to kinetic energy of the matter. In trying to come to a local measure of this energy we could look back at Chapter 8, Section 2 and consider the kinetic energy of the matter in a sphere, radius r , centred about a point. We could integrate this for $r = 0$ to $r = L$ where L is some arbitrary limit. However, no naturally occurring quantity suggests itself for L (The Planck length seems ridiculously small for the application), and thus the procedure is unappealing. Instead, the following approach might be taken.

Looking back to Section 3.1, the Universe was imagined, at some cosmic time, as a 3-dimensional hypersphere embedded in a Euclidean 4 space. Of course this can only model universes with global positive curvature ($k = +1$), but as the radius $R(t)$ increases the universe becomes more and more flat, locally.

As a start, imagine a 1-dimensional circle of mass density ρ (kg per co-moving length) expanding into a Euclidean 2 space. If the circle is of radius $R(t)$ then the kinetic energy of the mass per unit co-moving length is $\frac{1}{2}\rho\dot{R}^2(t)$, assuming $\dot{R}^2(t) \ll c$.

Similarly imagine a 2-dimensional sphere of mass density ρ (kg per co-moving area) expanding into a Euclidean 3 space. If the sphere is of radius

$R(t)$ then the kinetic energy of the mass per unit co-moving area is $\frac{1}{2}\rho\dot{R}^2(t)$, assuming $\dot{R}^2(t) \ll c$.

Lastly consider a 3-dimensional hypersphere of mass density ρ (kg per co-moving volume) expanding into a Euclidean 4 space. If the hypersphere is of “radius” $R(t)$ then by analogy with the 1 and 2 dimensional cases it might be asserted that it is valid to assign a kinetic energy to the mass per unit co-moving volume of $\frac{1}{2}\rho\dot{R}^2(t)$, again assuming $\dot{R}^2(t) \ll c$. It is not clear however how valid an assertion this is and in addition, the procedure suffers from the following problems:

In the $k = 1$ case, $R(t)$ has a simple interpretation as the radius of the hypersphere. However in the $k = 0$ spatially flat universe no such accessible view of $R(t)$ exists. The only meaningful quantities are $H(t) = \dot{R}(t)/R(t)$ and comparisons of $R(t)$ at different (cosmic) times, e.g. $R(\tau_1)/R(\tau_2)$ the ratio of $R(t)$ at $t = \tau_1$ to that at $t = \tau_2$. To use this concept in a spatially flat universe, it must be reformulated to remove the dependence on the absolute value of $R(t)$. Whilst this can be done in the two dimensional case, it does not “work out” neatly in the three dimensional case.

This approach, where we have begun to think about finding the potential by balancing changes in kinetic energy of matter with a change in gravitational potential energy, suffers from the difficulty of defining local kinetic energy when everything is co-moving and in an accelerating universe any kinetic term would be increasing as the potential term increased. Since acceleration, from recent evidence, seems the likely case we will leave this train of thought and start from the expected answer. If we assume that the cosmological redshift is due to an increasing potential then what can we deduce about it?

8.4 If cosmological redshift alters wavelength in proportion to changes in scale factor, what does this imply for the gravitational potential?

From the examination of textbook explanations of cosmological redshift, we would expect that if radiation leaves a source (in a Robertson Walker universe) with wavelength λ_{emit} and arrives at an observer with wavelength λ_{obs} , it might be expected that

$$\lambda_{\text{obs}}/\lambda_{\text{emit}} = R(\tau_{\text{obs}})/R(\tau_{\text{emit}}) = \nu_{\text{emit}}/\nu_{\text{obs}}$$

where ν_{emit} and ν_{obs} are the emitted and observed frequencies respectively. From the formula

$$\frac{\Delta\nu}{\nu_0} \simeq -\frac{gd}{c^2}$$

confirmed experimentally by Pound and Rebka, it might be expected that, replacing gd with $\Delta\Phi$

$$\nu_{\text{obs}} \simeq \nu_{\text{emit}}(1 - \Delta\Phi/c^2).$$

Therefore

$$\nu_{\text{obs}}/\nu_{\text{emit}} \simeq 1 - \Delta\Phi/c^2 = R(\tau_{\text{emit}})/R(\tau_{\text{obs}})$$

or

$$\Delta\Phi \simeq c^2(1 - R(\tau_{\text{emit}})/R(\tau_{\text{obs}}))$$

This is a sensible looking formula. When the time of emission and observation are close, $\Delta\Phi \rightarrow 0$. As the scale factors diverge, the potential difference rises asymptotically towards the value 1 in units where $c = 1$. This reflects the fact that as the density decreases the “next bit” of expansion takes less energy than the “last bit” as the distances between lumps of matter have increased and the “force” between them has decreased. Also as $\Delta\Phi \rightarrow 1$, $\nu_{\text{obs}} \rightarrow 0$, for any ν_{emit} (from $\nu_{\text{obs}}/\nu_{\text{emit}} = 1 - \Delta\Phi/c^2$ and putting $c = 1$).

This formula has the required “nice” result that there is no horizon where the recession velocity reaches c , as is the case in Doppler formulae.

If the scale factor of the universe at the time the observation is made (τ_{obs}) is set to the value 1, then the “cosmic potential” at the time of emission (as observed “now” at τ_{obs}) is

$$\Phi_C \simeq -c^2(1 - R(\tau_{\text{emit}}))$$

where $R(\tau_{\text{emit}})$ takes values from 0 (at the Big Bang) to 1 (now) and Φ_C goes from -1 to 0 in units where $c = 1$. Writing $R(\tau_{\text{emit}}) = a$, the spatial part of the Robertson Walker metric can thus be written as

$$(dl)^2 = a^2((dx)^2 + (dy)^2 + (dz)^2).$$

Re-arranging the above formula gives:

$$\Phi_C \simeq -c^2(1 - a)$$

or

$$\Phi_C \simeq (a - 1)$$

in units where $c = 1$.

It’s worth noting for later that therefore

$$a = \Phi_C + 1,$$

which means that

$$a^2 = (\Phi_C + 1)^2 \simeq 1 + 2\Phi_C$$

for small values of (the magnitude of) Φ_C . It remains now to show how this potential is related in either an approximate or exact way to the metric.

8.5 Gravitational potential and metric components

8.5.1 Weak field approximation.

As an example of a metric components being expressible in terms of a single potential, we can look at a “standard treatment” for the weak field approximation around a central gravitating mass where the appropriate space-time metric is the Schwarzschild metric. This gives a view of how metric components might be expressed as functions of a single gravitational potential in certain circumstances although it is not central to the main cosmological argument. The treatment below follows Kenyon [2].

In the case of weak, slowly varying gravitational fields the metric can be expressed as follows.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ defines the (small) deviations from the Minkowski metric, the magnitude of each of its components being of the order h , where $h \ll 1$.

Working in Cartesian coordinates where $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$, $\eta_{ij} = 0, i \neq j$. It is clear that weak, slowly changing fields are produced by stress-energy tensors dominated by the T_{00} component (i.e. the only significant source term is the energy density).

With zero cosmological constant, Einstein’s field equations are

$$G_{\alpha\beta} = (8\pi G/c^4)T_{\alpha\beta}$$

and

$$G_{\alpha\beta} = R_{\alpha\beta} - g_{\alpha\beta}R/2$$

thus

$$R_{\alpha\beta} - g_{\alpha\beta}R/2 = (8\pi G/c^4)T_{\alpha\beta}$$

or, contracting with $g^{\alpha\beta}$

$$R - g^\beta_\beta R/2 = (8\pi G/c^4)T^\beta_\beta$$

so as $g^\beta_\beta = 4$ (the number of dimensions)

$$R = -(8\pi G/c^4)T^\beta_\beta.$$

Substituting this back into the field equation above gives

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}(T_{\alpha\beta} - T^\beta_\beta g_{\alpha\beta}/2).$$

If T_{00} is the only significant component of the stress-energy tensor then this reduces to

$$R_{00} = \frac{8\pi G}{c^4}(T_{00} - T^0_0 g_{00}/2).$$

Now

$$2R_{\alpha\beta\gamma\delta} = g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta}$$

and since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, any derivative of $g_{\mu\nu}$ may be replaced by the same derivative of $h_{\mu\nu}$ as all derivatives of $\eta_{\mu\nu}$ are zero, then

$$2R_{\alpha\beta\gamma\delta} = h_{\alpha\delta,\beta\gamma} - h_{\beta\delta,\alpha\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta}.$$

Thus

$$2R_{00} = \eta^{\alpha\sigma}(h_{\sigma 0,0\alpha} - h_{00,\sigma\alpha} + h_{0\alpha,\sigma 0} - h_{\sigma\alpha,00})$$

where, as a linearizing approximation we have contracted the Riemann tensor with the Minkowski metric (as $g_{\mu\nu} \cong \eta_{\mu\nu}$).

In a slowly evolving or static system time derivatives $\rightarrow 0$. Thus

$$2R_{00} = \eta^{ij}h_{00,ij}$$

or $R_{00} = \frac{1}{2}h_{00,ii}$.

From the Schwarzschild metric

$$g_{\mu\nu} = \begin{pmatrix} 1 - 2GM/rc^2 & 0 & 0 & 0 \\ 0 & -(1 - 2GM/rc^2)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}$$

it is obvious that since $g_{00} = 1 - 2GM/rc^2 = \eta_{00} - h_{00} = 1 + h_{00}$ that

$$h_{00} = -2GM/rc^2$$

but since the classical potential $\Phi = -GM/r$ then

$$h_{00} = 2\Phi/c^2.$$

Now as $R_{00} = \frac{1}{2}h_{00,ii}$ it follows that

$$R_{00} = \Phi_{,ii}/c^2 = \nabla^2\Phi/c^2.$$

Turning to the stress-energy tensor, in a stationary system we expect the value of T_{00} at a point to be $= \rho c^2$.

Substituting these values (for R_{00} and T_{00}) into

$$R_{00} = \frac{8\pi G}{c^4}(T_{00} - T_0^0 g_{00}/2)$$

gives

$$\frac{\nabla^2\Phi}{c^2} = \frac{8\pi G}{c^4}(\rho c^2 - \rho c^2/2)$$

or

$$\nabla^2\Phi = 4\pi G\rho.$$

We can re-write this as

$$\nabla \cdot \nabla\Phi = 4\pi G\rho$$

but the force on a test mass m is $\mathbf{F}_m = (-\nabla\Phi)m$ so

$$\nabla \cdot \mathbf{F}_m = -4\pi mG\rho.$$

If we consider a small spherical volume V of radius r and surface area S over which ρ is constant, we can write

$$\int_S \mathbf{F} \cdot d\mathbf{S} = -4\pi mG \int_V \rho dV.$$

Putting $\int_V \rho dV = M$ we get

$$4\pi r^2 \mathbf{F}_m = -4\pi mGM$$

or

$$\mathbf{F}_m = -GMm/r^2$$

or Newton's law.

It's worth noting for later that it follows from the above, that the Schwarzschild metric tensor can be written in terms of the Newtonian gravitational potential as

$$g_{\mu\nu} = \begin{pmatrix} 1 - 2\Phi/c^2 & 0 & 0 & 0 \\ 0 & -(1 - 2\Phi/c^2)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}$$

or if the coordinate system used were Cartesian (centered on the gravitating mass) then the metric tensor would be

$$g_{\mu\nu} = \begin{pmatrix} 1 - 2\Phi/c^2 & 0 & 0 & 0 \\ 0 & -(1 - 2\Phi/c^2)^{-1} & 0 & 0 \\ 0 & 0 & -(1 - 2\Phi/c^2)^{-1} & 0 \\ 0 & 0 & 0 & -(1 - 2\Phi/c^2)^{-1} \end{pmatrix}.$$

Here we see that all the components of the metric tensor can be expressed in terms of a single quantity (Φ). This is as a result of the symmetry of the Schwarzschild metric. The Robertson Walker is also a highly symmetrical solution to Einstein's field equations and is also expressible in terms of a single quantity ($R(t)$). We may thus be able to rework it into a form where each component is a function of a single potential.

8.5.2 Can the components of the Robertson Walker metric be expressed in terms of the potential derived in 8.4 ?

The (spatially flat) Robertson Walker metric in Cartesian coordinates is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -R^2(t) & 0 & 0 \\ 0 & 0 & -R^2(t) & 0 \\ 0 & 0 & 0 & -R^2(t) \end{pmatrix}$$

or at time $t = \tau_{\text{emit}}$ with $a = R(\tau_{\text{emit}})/R(\tau_{\text{now}})$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix}$$

and thus

$$d\tau^2 = dt^2 - a^2(dx^2 + dy^2 + dz^2)$$

in units where $c = 1$.

In the above scheme t is treated such that in the absence of spatial displacements, $d\tau^2 = dt^2$. This means that any observer's notion of time dt wherever he or she may be in the universe (all points being equivalent) is equivalent to a notion of proper time $d\tau$ such as might be measured or obtained by looking at the average decay time of, say, stationary muons or the emission lines from stationary atoms anywhere in the universe (and thus at any relative scale factor compared to that at the time of observation).

The notion of "cosmic time" requires that all observers agree upon the time at which locally measurable quantities such as average density or CMBR temperature reach certain values but this does NOT require that $dt = d\tau$ in the absence of spatial displacements and is just as well satisfied by the following scheme:

Let dn (n for NOW) represent an infinitesimal passage of time experienced by an observer.

Put $d\tau = adn$ where $d\tau$ is the proper time as “seen” by, say, an atom at a relative scale factor a compared to the observer (a taking a value in the range $0 \rightarrow 1$).

An observer looking at emission lines from, say, a local hydrogen atom and from a cloud of hydrogen at a distance such that $a < 1$ sees less cycles of the radiated frequency from the remote hydrogen than from the local hydrogen in the time period δn by a factor a , such that if remote proper time is measured in cycles of this radiation and is compared to local time then $\delta\tau = a\delta n$. All other observers at the same distance from the remote hydrogen (same value of a) would arrive at the same value of δn , knowing a and observing $\delta\tau$. In this way the concept of “cosmic time” is maintained.

Since $d\tau = adn$ in the absence of spatial displacements then $d\tau^2 = a^2 dn^2$ and the metric equation can therefore be written as

$$d\tau^2 = a^2 dn^2 - a^2(dx^2 + dy^2 + dz^2)$$

and the metric tensor is thus

$$g_{\mu\nu} = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix}.$$

Clearly as $a \rightarrow 1$, $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ i.e. it is locally Minkowski and by substituting $dt = adn$ and $R(t) = a$ the original RW metric is regained thus proving that it is consistent with General Relativity.

In this form, cosmological redshift is a clear result of the metric equation reducing to $d\tau^2 = a^2 dn^2$ when the source and observer are both comoving.

Looking back to section 8.4 the relation between a and Φ_C was developed and it was shown that

$$a = (\Phi_C + 1).$$

The metric, in terms of the potential Φ_C is therefore

$$g_{\mu\nu} = \Phi_{\mu\nu} = \begin{pmatrix} (\Phi_C + 1)^2 & 0 & 0 & 0 \\ 0 & -(\Phi_C + 1)^2 & 0 & 0 \\ 0 & 0 & -(\Phi_C + 1)^2 & 0 \\ 0 & 0 & 0 & -(\Phi_C + 1)^2 \end{pmatrix}.$$

From this metric tensors it is clear that a co-moving observer seeing radiation from a co-moving source, observes time running more slowly at the source in an epoch when the cosmic potential was lower. This is the reason behind cosmological redshift, it is the Pound Rebka experiment on a cosmic scale!

8.5.3 What is the relationship between the cosmic potential and matter and the cosmological constant?

Firstly, what are the relationships between a , R , Φ_C and H ?

We know that $H = \dot{R}/R$ where R is the scale factor in the Robertson walker metric. In our new scheme where we have put $a = R/R_{\text{now}}$, we see that

$$H = \dot{a}.$$

Now we know that $a = \Phi_C + 1$ so $\dot{a} = \dot{\Phi}_C$ and therefore

$$H = \dot{\Phi}$$

And now - the relationship between Φ and ρ when $\Lambda = 0$

For a spatially flat universe we know that

$$\rho_0 = 3H^2/8\pi G$$

(where $c = 1$). Thus

$$(\dot{\Phi}_C)^2 = 8\pi G\rho_0/3.$$

And when $\Lambda \neq 0$, what then?

For a spatially flat universe we know that

$$3H^2 - \Lambda = 8\pi G\rho_0$$

(where $c = 1$). Thus

$$(\dot{\Phi}_C)^2 = \frac{1}{3}(8\pi G\rho_0 + \Lambda).$$

8.5.4 Notes on the constancy of the speed of light

We have, in previous sections, chosen to work in units such that the speed of light $c = 1$. The question arises, can we treat the speed of light c as a constant when space-time is described by the new metric $\Phi_{\mu\nu}$?

For any co-moving observer at any time and place, when performing local experiments, then $a \rightarrow 1$ and $\Phi_C \rightarrow 0$ and therefore,

$$\Phi_{\mu\nu} \rightarrow \eta_{\mu\nu}$$

and thus all co-moving observers will measure the same value for c in local experiments.

If, however a co-moving observer is looking at radiation from a far distant (co-moving) source, then that observer will conclude that

$$c_{\text{faraway}} = ac_{\text{local}} = (\Phi_C + 1)c_{\text{local}}.$$

This can be illustrated with a simple thought experiment.

Imagine a pulse of light repeatedly reflected between two mirrors A and B. A mechanism at A transmits a signal to a far distant (co-moving) observer every time the pulse arrives at A. An observer at A (also assumed to be co-moving, with distance AB relatively short and fixed) would see signals being transmitted at intervals of $2r/c$ seconds, where $r = (\text{distance } A \rightarrow B)$.

The distant observer would measure a time interval between the pulses of $2r/(\Phi_C + 1)c_{\text{local}}$ and would conclude that

$$c_{\text{faraway}} = ac_{\text{local}} = (\Phi_C + 1)c_{\text{local}}$$

as previously stated. Thus the observer would conclude that the speed of light in the distant co-moving system was slower than that observed locally.

Now consider the same experimental arrangement in the Schwarzschild space-time around a black hole. If the mirrors were near (but outside) the event horizon and the observer were also in free fall at a large distance from the black hole, then the observation and conclusion would be the same. We do not, however have any difficulty in expressing the components of Schwarzschild metric in units such that $c = 1$ and neither therefore should it be a problem to express the components of $\Phi_{\mu\nu}$ in such units. It must be remembered that c is the value that all co-moving observers will obtain for the speed of light.

8.5.5 Is Φ_C really a gravitational potential? - a quick “sanity check”

In the metric, $g_{\mu\nu} = \Phi_{\mu\nu}$, g_{00} , being the ratio of increments in remote proper time to increments in local observers time, is dimensionless.

Now

$$g_{00} \simeq (1 + \frac{2\Phi_C}{c^2})$$

in the weak field case and in the strong field,

$$g_{00} = (\frac{\Phi_C}{c^2} + 1)^2.$$

In either case, therefore, the units of Φ_C in S.I. units must be m^2s^{-2}

In Newtonian gravity, the units of G are $\text{Nm}^2\text{Kg}^{-2}$ (from $F = GMm/r^2$). The gravitational potential due to a central gravitating mass M is

$$\Phi = GM/r$$

and it therefore has units of

$$(\text{Nm}^2\text{Kg}^{-2})(\text{Kgm}^{-1})$$

or Nm/Kg or J/Kg as expected.

Noting that in Special Relativity, energy $E = mc^2$, we can write the units of energy (normally joules J) as $\text{Kgm}^2\text{s}^{-2}$. The units of the Newtonian potential Φ can therefore also be expressed as

$$\text{Kgm}^2\text{s}^{-2}\text{Kg}^{-1}$$

or

$$\text{m}^2\text{s}^{-2}$$

just as they are for Φ_C . Equally, therefore the units of Φ_C may be expressed as J/Kg . It is therefore dimensionally consistent to think of Φ_C as a gravitational potential.

8.5.6 Potentials in General relativity

In Chapter 8 section 5.1 we showed that in the case slowly varying gravitational fields (and with test particles with velocities very much less than the speed of light), that General Relativity reduces to Newton's law of gravity. We showed that the metric components were expressible in terms of the Newtonian gravitational potential only. Kenyon [2] page 86 table 7.1 gives a table of "GR quantities" with their "Newtonian analogues", in which the metric components are analogous to the potential, the metric connections (being sums of the first derivatives of the components) are analogous to the gravitational force and the components of the Riemann tensor (being sums of the second derivatives of the metric components) are analogous to tidal forces in Newtonian physics.

We have also seen how the static Schwarzschild metric of the space-time outside a spherical gravitating mass may be expressed in terms of the single

scalar potential. We now must investigate the reasonableness of the expressing the components of the modified Robertson Walker metric in terms of a single scalar potential.

Static fields

We start by noting that for a field to be expressed in terms of a single scalar potential the change in energy of a test particle in moving from one point to another (no other forces being applied to it - other than those due to the gradient of the potential in question!) should be independent of the path taken. Thus a test mass moving from point A to point B in a Newtonian gravitational field will always gain or lose the same amount of energy. We would therefore expect that the test particle moving from A to B and then back to A by any path would neither gain nor lose energy. Such a field is known as a “conservative” field. Allowing the rest mass of the test particle to reduce to zero we would expect the change in energy of photons leaving A and arriving at B to be independent of the path taken in any conservative field. We can therefore devise an experiment (a thought experiment) to test that changes in energy are independent of path.

The thought experiment

In this experiment, we have a source consisting of a laser and beam splitter and two mirrors arranged to reflect both beams to an observer where the frequencies of the beams can be compared (and thus the photon energies).

Accepting that the photons neither gain nor lose energy on reflection by the mirrors and that the observer, source and mirrors remain a fixed proper distance each from every other part of the experiment, this apparatus may be used to test the properties of a static field. If for instance the apparatus was a fixed distance from a non rotating spherical mass (say on the surface

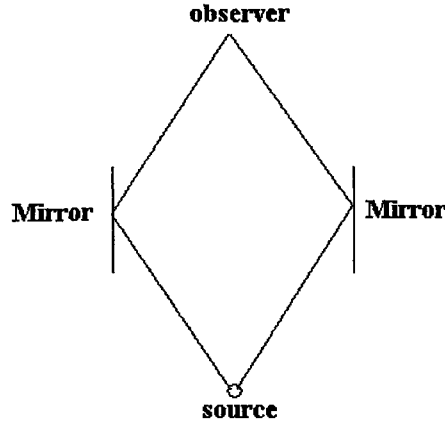


Figure 8.1 Apparatus for thought experiment

of the Earth, ignoring the rather slow spin of the Earth) then, whatever the orientation of the apparatus we would expect that the observer would note that the photon energies were the same for either route through the apparatus. If this were not the case then a light beam sent round the loop “source-mirror-observer-other mirror-source” would either gain or lose energy and we could construct a “free energy” source on earth! This is consistent with the idea that Schwarzschild space-time surrounding a non rotating spherical mass is describable in terms of a single scalar potential and we know that particles in free fall follow orbits of constant energy although not quite those of Newtonian physics. Schutz [3] chapter 11 derives effective potentials for orbits of both massive and massless particles (photons) in Schwarzschild geometry.

Non static fields

Our modified Robertson Walker metric is not static. The metric components are defined in terms of the time dependant scale factor. In general, non static geometry implies trajectories for test particles that are not of constant

energy. Since gravitational effects (disturbances in the geometry) in General Relativity travel at the speed of light, non static solutions usually involve the transfer of gravitational energy from one place to another by gravitational waves. That these waves carry energy is demonstrated in, for instance, Schutz [3] Ch 9, Section 4 or Kenyon [2] Ch 10 Section 2 P133. This energy can be imparted to test particles. If this were not the case then there would be no way to make a gravitational wave detector. In our thought experiment, in the presence of a non static gravitational field with disturbances transmitted at the speed of light, the photon energies will in general depend on the path taken and we can't therefore define the field in terms of a single scalar potential.

Why can the modified Robertson Walker metric be described in terms of a single scalar potential?

In a universe of fine evenly distributed dust that is isotropic and homogeneous we might suppose that the local geometry depends on the local matter density "now" and the density of more distant matter at an earlier cosmic time, as any influence such distant matter has on the local geometry travels at the speed of light. The influence of all such matter being combined (we do not have to suppose any particular law of combination of such influences, such as linear superposition, neither do we have to suppose that the speed of light is any particular value or even that it is constant - it could vary with scale factor for instance) to result in the local geometry observed. There are, however, no preferred points and no preferred directions in this homogeneous and isotropic universe so what is true of one point at a particular cosmic time is equally true of any other at the same cosmic time. So the influences of all matter near and far, at any point at a particular time is the same as that of all matter at all others at the same cosmic time and since there are no special directions in space, the local geometry can be characterized by one number, a scalar. This scalar is the scale factor $R = R(t)$.

In an isotropic, homogeneous, dust universe, we would certainly find that

the observed energy of the photons from a source emitted at one cosmic time and arriving at an observer at another was independent of path (since the universe would look the same after any rotation or spatial translation) all such paths being equivalent. In a spatially flat, perfectly uniform universe, however, there is only one such path (geodesic) between any two events. However, in the actual Universe, if the two images of, say, a quasar were to be observed due to the presence of a galaxy on the line of sight between the quasar and the observer, acting as a gravitational lens, then we should expect that the images would both show the same redshift as long as the cosmic “time of flight” for photons following each path was not significantly different. Therefore, when considering the radiation from distant co-moving sources, any change can only be due simply to the difference in the one scalar value characterizing local geometry, from one cosmic time (time of emission) to another (time of observation). This being the case, we would observe the required independence of path and this would allow us to assert that the cosmological redshift (for objects far enough away that the universe is essentially homogeneous at the scale of this distance) was due to a change in cosmic gravitational potential, this potential being expressed in terms of the scale factor $R(t)$.

A more detailed look at symmetry follows in the next chapter that arrives at formal results consistent with the above discussion.

Chapter 9

Killing vectors, symmetry and potential

9.1 A quick look at symmetry

A space may be said to possess a symmetry if there is a coordinate system in which the components of the metric $g_{\mu\nu}$ are independent of one or more of the coordinates. In the metric we derived in the previous chapter (Chapter 8)

$$\Phi_{\mu\nu} = \begin{pmatrix} (\Phi_C + 1)^2 & 0 & 0 & 0 \\ 0 & -(\Phi_C + 1)^2 & 0 & 0 \\ 0 & 0 & -(\Phi_C + 1)^2 & 0 \\ 0 & 0 & 0 & -(\Phi_C + 1)^2 \end{pmatrix}$$

we can see that the components are independent of all the coordinates (r, θ, ϕ) or (x, y, z) except the time coordinate, since Φ_C is a function of time. In contrast the Schwarzschild metric is independent of t and ϕ and is “stationary and axisymmetric”. However, the space time around a non-rotating central gravitating mass can be described by any number of different Schwarzschild metrics, each with an axis of symmetry and none of these axes being the same as any other. Thus the axisymmetric nature of the Schwarzschild solution is not a property of the space-time itself, just of the coordinate system used

to describe it. The above metric, $\Phi_{\mu\nu}$, is obviously not static but has more spatial symmetry than the Schwarzschild metric.

As can be seen from the Schwarzschild metric, this way of looking at symmetry is coordinate system dependant. We will now look at a coordinate-system-independent way of expressing the symmetries.

9.2 Tangent vectors in metric spaces

If, in a metric space, we can identify a continuous set of real numbers x^i with the coordinates of points in the space then, when we join any set of points with a curve governed by some parameter λ , we will find that the derivatives $dx^i/d\lambda$ exist.

In general, we often express a vector \mathbf{v} in Euclidean 3-space as the weighted sum of a set of basis vectors \mathbf{e}^i . Thus

$$\mathbf{v} = \sum_i a^i \mathbf{e}^i$$

In our metric space we can define a tangent vector $d/d\lambda$ as

$$\frac{d}{d\lambda} = \sum_i \frac{dx^i}{d\lambda} \frac{\partial}{\partial x^i}$$

where, in comparing the above two equation, we see that the $\partial/\partial x^i$ form a basis for the vector and the coordinate derivatives, the coefficients. To see how this works we might imagine the curve (parameterized by λ) in Euclidean 3-space in say, spherical; polar coordinates $r, \theta, \phi \equiv x^i, i = 1, 2, 3$, then the change $d\mathbf{S}$ in position vector \mathbf{S} with change in parameter λ along the curve would be

$$\frac{d\mathbf{S}}{d\lambda} = \sum_i \frac{dx^i}{d\lambda} \frac{\partial \mathbf{S}}{\partial x^i}.$$

and for instance the r -component of the tangent vector to the curve would be

$$\frac{dr}{d\lambda} = \sum_i \frac{dx^i}{d\lambda} \frac{\partial r}{\partial x^i}.$$

where here r represents the r component of \mathbf{S} . In space-time the curve in question might be the world-line of a test particle. If we parametrize the curve with the particle's proper time τ , the tangent vector is

$$\frac{d}{d\tau} = \sum_{\nu} \frac{dx^{\nu}}{d\tau} \frac{\partial}{\partial x^{\nu}} \quad (9.1)$$

and we see that the coefficients $dx^{\nu}/d\tau$ are the magnitudes of the components of the 4-velocity of the particle in the basis $\partial/\partial x^{\nu}$. Thus $d/d\tau$ represents the particle's 4-velocity, u^{ν} along the curve

$$\frac{d}{d\tau} \equiv u^{\nu}$$

which is equivalent to saying $d\mathbf{S}/d\tau = \mathbf{U}$ in Euclidean space, or if an event on the world line of a particle is expressed as the 4-position s^{ν} then the equation becomes $ds^{\nu}/d\tau = u^{\nu}$.

Now the 4-momentum p^{ν} of a particle of mass μ is given by $p^{\nu} = \mu u^{\nu} = \mu(d/d\tau)$. If we put $\lambda = \tau/\mu$ then we see that

$$\frac{d}{d\lambda} \equiv p^{\nu}$$

(equivalent to saying $d\mathbf{S}/d\lambda = \mathbf{P}$ in Euclidean space) and if, for instance, we wanted to find the r -component of the 4-momentum of a particle on this curve we would find

$$\frac{dr}{d\lambda} = \sum_{\nu} \frac{dx^{\nu}}{d\lambda} \frac{\partial r}{\partial x^{\nu}}.$$

where again r is the r component of the 4-position vector.

These ideas are used in the next section on Killing vectors.

9.3 Killing vectors - a coordinate system independent way of looking at symmetry.

If $g_{\mu\nu}$ in a particular coordinate system x^{α} is independent of one of the coordinates x^K such that

$$\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} = 0$$

for

$$\alpha = K$$

then

$$\frac{\partial g_{\mu\nu}}{\partial x^K} = 0.$$

Following the argument in [34] page 650 we can say that any curve, $x^\alpha = c^\alpha(\lambda)$ (where the curve is parameterized by λ) can be translated in the x^K direction by a coordinate shift $\Delta x^K = \varepsilon$ to form a “congruent” or equivalent curve defined by

$$x^\alpha = c^\alpha(\lambda)$$

for $\alpha \neq K$ and

$$x^K = c^K(\lambda) + \varepsilon$$

(all values of K components shifted by ε).

The length of any curve parameterized by λ between two points for which $\lambda = \lambda_1$ and $\lambda = \lambda_2$ is given by

$$L = \int_{\lambda_2}^{\lambda_1} [g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}]^{\frac{1}{2}} d\lambda.$$

In the shifted curve case

$$g_{\mu\nu}(x(\lambda)) \rightarrow g_{\mu\nu}(x(\lambda)) + \varepsilon \frac{\partial g_{\mu\nu}}{\partial x^K}$$

so

$$L(\varepsilon) = \int_{\lambda_2}^{\lambda_1} [(g_{\mu\nu}(x(\lambda)) + \varepsilon \frac{\partial g_{\mu\nu}}{\partial x^K}) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}]^{\frac{1}{2}} d\lambda$$

but as we have seen, $\partial g_{\mu\nu} / \partial x^K = 0$ so,

$$L(\varepsilon) = \int_{\lambda_2}^{\lambda_1} [(g_{\mu\nu}(x(\lambda)) + \varepsilon \frac{\partial g_{\mu\nu}}{\partial x^K}) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}]^{\frac{1}{2}} d\lambda = \int_{\lambda_2}^{\lambda_1} [g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}]^{\frac{1}{2}} d\lambda = L.$$

Therefore we can write

$$\frac{dL}{d\varepsilon} = 0 \tag{9.2}$$

we now define

$$\xi \equiv \frac{d}{d\varepsilon} = \frac{\partial}{\partial x^K} \tag{9.3}$$

which is known as a Killing vector. In equation 9.3, $\partial/\partial x^K$ is of course one of the basis vectors from equation 9.1, in the case of this “preferred” frame where $\partial g_{\mu\nu}/\partial x^K = 0$. If we have a series of congruent curves parameterized by λ then we would find, say, the r component of the position vector, $dr/d\lambda$ to one curve was equal to $dr'/d\lambda$ on the next if the next curve is just shifted from the first by distance $d\varepsilon$ in the x^K direction. This is equally true of other component and of course leads to the result, equation 9.2. This is an example of isometry.

ξ provides an “infinitesimal description of these length preserving properties”. Killing vectors act on the components of the position vector along a curve (usually a geodesic) and yield information on velocity or momentum depending on the parameterization of the curve. We will see that it satisfies Killing’s equation

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$$

which shows that its covariant derivative is antisymmetric in the two labels μ, ν .

To prove this, it is only necessary to show that it holds in one preferred frame for it to hold in all, as it is expressed in covariant form (i.e. it is a tensor equation).

In a “preferred” coordinate system, the vector field has components

$$\xi^\mu = \delta_K^\mu$$

which we can see from

$$\xi \equiv \frac{d}{d\varepsilon} = \frac{\partial}{\partial x^K}$$

so the covariant derivative of this vector has components

$$\xi_{\mu;\nu} = g_{\mu\alpha} \xi_{;\nu}^\alpha = g_{\mu\alpha} \left(\frac{\partial \xi^\alpha}{\partial x^\nu} + \Gamma_{\nu\sigma}^\alpha \xi^\sigma \right).$$

Using $\xi \equiv d/d\varepsilon = \partial/\partial x^K$ and $\xi^\mu = \delta_K^\mu$ we see that

$$\xi_{\mu;\nu} = g_{\mu\alpha} \left(\frac{\partial \delta_K^\alpha}{\partial x^\nu} + \Gamma_{\nu\sigma}^\alpha \delta_K^\sigma \right) = 0 + g_{\mu\alpha} \Gamma_{\nu K}^\alpha = \Gamma_{\mu\nu K} = \frac{1}{2} (g_{\mu K, \nu} + g_{\mu\nu, K} - g_{\nu K, \mu})$$

but

$$g_{\mu\nu,K} = \frac{\partial g_{\mu\nu}}{\partial x^K} = 0$$

(since we know that this is the symmetry property of the metric in this frame). So

$$\xi_{\mu;\nu} = \frac{1}{2}(g_{\mu K,\nu} - g_{\nu K,\mu})$$

which is antisymmetric and obeys Killing's equation as required.

One of the things that makes Killing vectors useful is the fact that in any space-time that has a symmetry described by a Killing vector field, looking at motion along any geodesic the scalar product of the tangent vector with the Killing vector is a constant.

$$p_K = \mathbf{p} \cdot \boldsymbol{\xi} = \text{constant}.$$

We can see this by looking at the coordinate system in which we can write $\boldsymbol{\xi} = \partial/\partial x^K$. The scalar product becomes

$$\mathbf{p} \cdot \boldsymbol{\xi} = p_\alpha \xi^\alpha = p_\alpha \delta_K^\alpha = p_K = \text{constant}.$$

Therefore the symmetry guarantees the conservation of the K th component of the momentum.

9.4 Conserved quantities in Minkowski geometry

The Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}$$

(in spherical polar coordinates) is axi-symmetric. Looking at the diagram below, Fig 9.1, it is clear that the length of the line a b is unaffected by the

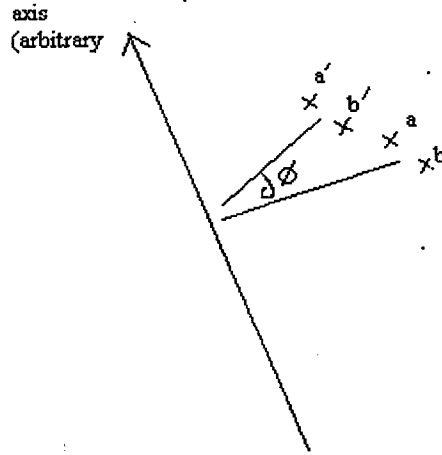


Figure 9.1 Displacement in Minkowski Space-time

elemental displacement $\varepsilon \frac{\partial}{\partial \phi}$. This is of course a coordinate system dependant definition, but clearly the actual displacement still preserves the length a b expressed in any other coordinate system and can therefore be written in a coordinate independent way as $\varepsilon \xi$ where ξ is a Killing vector. Thus

$$\varepsilon \xi = \varepsilon \frac{\partial}{\partial \phi}.$$

We have seen that, in general, when considering motion along a geodesic, the tangent vector

$$\mathbf{p} \cdot \xi = p_\alpha \xi^\alpha = p_\alpha \delta_K^\alpha = p_K = \text{constant}$$

and thus in Minkowski space-time, we see that $\mathbf{p} \cdot \xi = p_\phi = \text{constant}$. This is a statement of the conservation of angular momentum. Similarly, since the components of the metric are independent of time t so we can show that

$$p_0 = p_t = \text{constant}.$$

This is a statement of the conservation of energy.

9.5 Application of these ideas to the non static modified Robertson Walker space-time

If we consider motion along null geodesics in Minkowski space-time, we have seen that energy is conserved. Thus for photons we expect $E = \hbar\omega =$ constant and that therefore the frequency of radiation as it propagates will be constant.

In the Robertson Walker picture where, if we write $a = R(\tau_{\text{emit}})/R_{\text{NOW}}$, the metric is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix}.$$

We can see that g_{11} , g_{22} , and g_{33} are all functions of time and we know, therefore, that in general $p_0 \neq$ constant. However for an observer performing local experiments, then $a \rightarrow 1$ and therefore, $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ and so in accordance with common experience we find that when observing light from nearby sources, energy and thus the frequency of radiation is conserved.

If, however, we are looking at the radiation from distant astronomical sources then we have already seen, in chapter 7, section 2, that frequencies are “cosmologically redshifted” as the scale factor varies, and it follows directly from the argument in chapter 7, section 2, that $a = \omega_{\text{obs}}/\omega_{\text{emit}}$ or $a\omega_{\text{emit}} = \omega_{\text{obs}}$.

In our new scheme, we think of the observed frequency as being measured in terms of radians per unit coordinate time n . We can write

$$\omega_{\text{obs}} = 2\pi/\delta n$$

where δn is the (assumed small) period of oscillation of the observed radiation.

The emitted frequency is thought of as being measured in terms of the proper time of the source τ . We can write

$$\omega_{\text{emit}} = 2\pi/\delta\tau.$$

Now for co-moving source and observer, we know that $d\tau = a\delta n$ and thus $\delta\tau = a\delta n$ for small but finite $\delta\tau$ and δn . So

$$\omega_{\text{emit}} = 2\pi/\delta\tau = 2\pi/a\delta n.$$

We define $\omega'_{\text{emit}} = 2\pi/\delta n$ and we see that

$$\omega'_{\text{emit}} = a\omega_{\text{emit}}$$

(or equivalently $\omega'_{\text{emit}} = (1 + \Phi_C)\omega_{\text{emit}}$) and since $a\omega_{\text{emit}} = \omega_{\text{obs}}$ we get

$$\omega'_{\text{emit}} = \omega_{\text{obs}}$$

or

$$E'_{\text{emit}} = E_{\text{obs}}.$$

The choice of time coordinate n and of $g_{00}(n) = a(n) = (1 + \Phi_C)$ as opposed to the Robertson Walker choice of t and $g_{00} = 1$, in chapter 8 section 5 may have seemed fairly arbitrary. We could come up with any number of different time like coordinates and one of these could have been chosen, just to give the expected “right” answer, but we now see that this is the only choice that preserves the photon energy and it is the view of the author that this choice is fundamentally singled out since it requires no energy loss from the photons and thus removes the quantum mechanical difficulties that cosmological redshift would otherwise encounter (see chapter 7, section 3).

Chapter 10

Quantum field theory in an expanding universe. - A quantum problem deserves a quantum answer.

10.1 Space-time in quantum theory

Generally, Quantum Field theory (QFT), is a (set of) theory(s) that is(are) formulated in a background space-time, this being the Minkowski space-time of Special Relativity. This certainly applies to Quantum Electro Dynamics (QED). Some attempts at a “background less” quantum theory of gravity (loop quantum gravity) are described by Smolin [35] Ch 10. We will limit ourselves to looking for the background space-time that preserves the structure of “standard” QFT but results in the expected cosmological redshift. We will start this process by looking at a simple quantum harmonic oscillator and will examine it when formulated in our new time coordinate. We then demonstrate that any mode of the classical electromagnetic field in an arbitrarily large cavity, is equivalent to a harmonic oscillator. Using this result we then quantize the field in the cavity. We again formulate it in the new coordinates and show how cosmological redshift is included. This whole chapter follows arguments presented in “The Quantum Vacuum” by Peter W. Miloni, [36], “The Quantum Theory of Light” by Rodney Loudon, [37] and “Gauge Field Theories, An Introduction with Applications” by Mike

Guidry [38].

10.2 First a quantum harmonic oscillator

The classical Hamiltonian for a one dimensional harmonic oscillator is

$$H = (p^2/2m) + \frac{1}{2}m\omega^2 q^2$$

where q is position and p momentum.

The equivalent quantum mechanical formulation is

$$\hat{H} = (\hat{p}^2/2m) + \frac{1}{2}m\omega^2 \hat{q}^2$$

where \hat{H} , \hat{p} and \hat{q} are now operators (such that for instance $\hat{p}|x\rangle$ where \hat{p} yields information on momentum when operating on the state function $|x\rangle$). \hat{p} and \hat{q} obey the commutation relation

$$[\hat{q}, \hat{p}] = i\hbar.$$

We define the further operators

$$\hat{a} = (2m\hbar\omega)^{-1/2}(m\omega\hat{q} + i\hat{p})$$

and

$$\hat{a}^\dagger = (2m\hbar\omega)^{-1/2}(m\omega\hat{q} - i\hat{p}).$$

Clearly

$$\hat{a}\hat{a}^\dagger = (2m\hbar\omega)^{-1}(\hat{p}^2 + m^2\omega^2\hat{q}^2 - im\omega\hat{q}\hat{p} + im\omega\hat{p}\hat{q})$$

or

$$\hat{a}\hat{a}^\dagger = (\hbar\omega)^{-1}(\hat{H} + \frac{1}{2}\hbar\omega)$$

and similarly

$$\hat{a}^\dagger\hat{a} = (\hbar\omega)^{-1}(\hat{H} - \frac{1}{2}\hbar\omega).$$

Subtracting the two equations above gives the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

and adding them gives an expression for \hat{H} .

$$\hat{H} = \frac{1}{2}\hbar\omega(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}).$$

If $|n\rangle$ is an energy eigenstate of the oscillator and E_n is the corresponding energy eigenvalue, then the eigenvalue equation is

$$\hat{H}|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})|n\rangle = E_n|n\rangle.$$

Multiplying through by \hat{a}^\dagger gives

$$\hat{H}\hat{a}^\dagger|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{a}^\dagger)|n\rangle = E_n\hat{a}^\dagger|n\rangle.$$

Using

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

we see that

$$\hat{a}^\dagger\hat{a}^\dagger\hat{a} = \hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger$$

so

$$\hbar\omega(\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger + \frac{1}{2}\hat{a}^\dagger)|n\rangle = E_n\hat{a}^\dagger|n\rangle$$

or

$$\hat{H}\hat{a}^\dagger|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} - 1 + \frac{1}{2}\hat{a}^\dagger)\hat{a}^\dagger|n\rangle = E_n\hat{a}^\dagger|n\rangle$$

or

$$\hat{H}\hat{a}^\dagger|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{a}^\dagger)\hat{a}^\dagger|n\rangle = (E_n + \hbar\omega)\hat{a}^\dagger|n\rangle.$$

This is a new eigenvalue equation with eigenstate $\hat{a}^\dagger|n\rangle$ and energy eigenvalue $(E_n + \hbar\omega)$. We can write the new state in terms of the old as

$$|n+1\rangle = \hat{a}^\dagger|n\rangle$$

with the new energy eigenvalue

$$E_{n+1} = E_n + \hbar\omega$$

and \hat{a}^\dagger is known as the creation operator, as repeated application creates another quantum of energy in the oscillator.

Similarly we can show that

$$\hat{H}\hat{a}|n\rangle = (E_n - \hbar\omega)\hat{a}|n\rangle.$$

The new state

$$\hat{a}|n\rangle = |n-1\rangle$$

has energy eigenvalue

$$E_{n-1} = E_n - \hbar\omega$$

and \hat{a} is consequently known as the destruction operator. Repeated application of the destruction operator removes quanta of energy from the system.

Unlike the creation operator, we cannot repeatedly apply \hat{a} an arbitrarily large number of times as at some point we reach the ground state of the system.

Writing $\hat{n} = \hat{a}^\dagger \hat{a}$ where \hat{n} is the number operator with eigenvalue equation $\hat{n}|n\rangle = n|n\rangle$. The original eigenvalue equation becomes

$$\hat{H}|n\rangle = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})|n\rangle = E_n|n\rangle$$

or

$$E_n|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle$$

then the ground state is given by

$$E_0|0\rangle = \hbar\omega(\frac{1}{2})|0\rangle$$

or

$$E_0 = \frac{1}{2}\hbar\omega$$

and in general

$$E_n = (n + \frac{1}{2})\hbar\omega, \quad n = 0, 1, 2, \dots$$

10.3 Normalized form of the equations

In the following sections on field theory, the normalized form of the above relations is used, so we will now put the above relations into the normalized form.

Earlier we wrote

$$|n+1\rangle = \hat{a}^\dagger |n\rangle$$

and

$$|n-1\rangle = \hat{a} |n\rangle.$$

The normalization conditions are

$$\langle n-1|n-1\rangle = \langle n|n\rangle = \langle n+1|n+1\rangle = 1.$$

The normalized version of the equations include additional terms thus

$$|n+1\rangle = \hat{a}^\dagger |n\rangle$$

becomes

$$C_1 |n+1\rangle = \hat{a}^\dagger |n\rangle$$

and

$$|n-1\rangle = \hat{a} |n\rangle$$

becomes

$$C_2 |n-1\rangle = \hat{a} |n\rangle.$$

So,

$$(C_2)^2 \langle n-1|n-1\rangle = \hat{a} \hat{a}^\dagger \langle n|n\rangle$$

and

$$|C_2|^2 = n$$

or

$$C_2 = n^{\frac{1}{2}}.$$

Similarly, as

$$[\hat{a}, \hat{a}^\dagger] = \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1$$

so

$$\hat{a}\hat{a}^\dagger = n + 1.$$

So,

$$(C_1)^2 = \langle n + 1 | n + 1 \rangle = \hat{a}\hat{a}^\dagger \langle n | n \rangle$$

and

$$|C_1|^2 = n + 1$$

or

$$C_1 = (n + 1)^{\frac{1}{2}}$$

and the normalized form of the equations becomes

$$(n + 1)^{\frac{1}{2}} |n + 1\rangle = \hat{a}^\dagger |n\rangle$$

and

$$n^{\frac{1}{2}} |n - 1\rangle = \hat{a} |n\rangle.$$

This is the form in which these relations are used later.

10.4 The harmonic oscillator in an expanding universe

The analysis in the section 10.2, tacitly assumed a flat “classical” view of space and time and the time coordinate is the proper time of the oscillator. We now have to look at the harmonic oscillator from the point of view of a remote observer in an expanding universe.

An observer might deduce information about the changes in state of a remote harmonic oscillator by observing the radiation emitted when it changes state. Using our previously defined view of emitted frequency as seen in the observer’s frame, where energy of the radiation is conserved we see that the energy eigenvalues of the oscillator would be

$$E_n = (n + \frac{1}{2})\hbar\omega'_{\text{emit}}, \quad n = 0, 1, 2, \dots$$

where

$$\omega'_{\text{emit}} = a\omega_{\text{emit}} = (\Phi_C + 1)\omega_{\text{emit}} = \omega_{\text{obs}}.$$

Thus, the energy levels available as seen by the observer are

$$E_n = (n + \frac{1}{2})\hbar(\Phi_C + 1)\omega_{\text{emit}}, \quad n = 0, 1, 2, \dots$$

and the ground state is given by

$$E_0 = \frac{1}{2}\hbar(\Phi_C + 1)\omega_{\text{emit}}.$$

It is clear from the above that the theory developed in Section 10.2 for the quantum harmonic oscillator correctly describes the oscillator in an expanding universe if we replace ω everywhere it occurs with $\omega' = a\omega = (\Phi_C + 1)\omega$. We will now see how this idea is carried over to a quantum field.

10.5 A field mode as a harmonic oscillator

We start with Maxwell's equations for a free field, i.e. where there are no source terms and consider such a field in a cavity that imposes periodic boundary conditions. We show that each field mode is a simple harmonic oscillator. Maxwell's equations for the free field (where there are no source terms) are

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}.$$

As $\nabla \cdot \nabla \times \{\text{any vector}\} = 0$ we can define \mathbf{A} from

$$\mathbf{B} = \nabla \times \mathbf{A}$$

and thus $\nabla \cdot \mathbf{B} = 0$ is automatically satisfied.

From $\nabla \times \mathbf{E} = -\partial \mathbf{B}/c\partial t$ we get

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{A})$$

or

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

(In general we would find $\mathbf{E} = -\partial \mathbf{A}/c\partial t - \nabla \phi$ but we are considering the free field case where there are no sources (charge or current) and thus the scalar potential $\phi = \nabla \phi = 0$).

Using $\nabla \times \mathbf{B} = \partial \mathbf{E}/c\partial t$ we get

$$\nabla \times \nabla \times \mathbf{A} = \frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right)$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}.$$

The electromagnetic field is said to be in the “Coulomb gauge” when the vector potential satisfies the condition $\nabla \cdot \mathbf{A} = 0$. [36]

(A note on gauge fixing - Any particular choice of E and B , can be defined in terms on any number of pairs of potentials, A and ϕ . Because $\nabla \times \nabla \phi = 0$, we can chose $\dot{A} \equiv A + \nabla \chi$ (where χ is an arbitrary scalar function) and arrive at the same B as $B = \nabla \times A = \nabla \times (A + \nabla \chi) = \nabla \times \dot{A}$. To keep the same E we must change ϕ so that $\phi \rightarrow \dot{\phi} \equiv \phi - \partial \chi / \partial t$. These “Gauge” transformations leave the Maxwell equations invariant and this allows us to chose a set of potentials (A, ϕ) such that $\nabla \cdot A + \partial \phi / \partial t = 0$. We therefore get the decoupled equations

$$\nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -j$$

and

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\rho$$

which we can solve separately for A and ϕ . This is known as the Lorentz gauge. When ρ and j are zero and we can set $\phi = \partial \phi / \partial t = 0$, then we are considering the radiation field in which B and E are orthogonal and transverse to the wave propagation, we are then working in the radiation or Coulomb gauge defined by $\nabla \cdot A = 0$ [38])

So we arrive at the wave equation in vector potential

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0.$$

We are eventually going to quantize the field by the replacement of the classical vector potential \mathbf{A} by a quantum mechanical operator $\hat{\mathbf{A}}$. To do this in as simple a way as possible [37], we consider the field to be in a cubic region of space of side L and volume $V = L^3$. This cube imposes periodic boundary conditions on the solutions and we arrive at a discrete set of (sinusoidal) solutions

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{K}} \sum_{\lambda=1,2} \mathbf{e}_{\mathbf{K}\lambda} A_{\mathbf{K}\lambda}(\mathbf{r}, t)$$

where,

$$A_{\mathbf{K}\lambda}(\mathbf{r}, t) = A_{\mathbf{K}\lambda}(t) e^{i\mathbf{K}\cdot\mathbf{r}} + A_{\mathbf{K}\lambda}^*(t) e^{-i\mathbf{K}\cdot\mathbf{r}}.$$

Of course as L is made very big we would approach a continuous spectrum of allowed sinusoidal solutions and thus any general solution could be formed as the weighted sum of such solutions. The components of the wavevector \mathbf{K} take the values

$$K_x = 2\pi\nu_x/L, \quad K_y = 2\pi\nu_y/L, \quad K_z = 2\pi\nu_z/L.$$

$$\nu_x, \nu_y, \nu_z = 0, \pm 1, \pm 2, \dots$$

λ , represents the polarization state and thus $\mathbf{e}_{\mathbf{K}\lambda}$ are the unit polarization vectors. The Coulomb gauge condition is satisfied if these are both transverse (λ only takes the values 1 or 2 see below), with

$$\mathbf{e}_{\mathbf{K}\lambda} \cdot \mathbf{K} = 0.$$

Choosing the polarization vectors to be perpendicular to each other gives $\mathbf{e}_{\mathbf{K}1} \cdot \mathbf{e}_{\mathbf{K}2} = 0$ and $\mathbf{e}_{\mathbf{K}1} \cdot \mathbf{e}_{\mathbf{K}1} = 1$ which we can combine as $\mathbf{e}_{\mathbf{K}\lambda} \cdot \mathbf{e}_{\mathbf{K}\lambda'} = \delta_{\lambda, \lambda'}$.

The modal components of \mathbf{A} , separately obey the field equation

$$\nabla^2 A_{\mathbf{K}\lambda} - \frac{1}{c^2} \frac{\partial^2 A_{\mathbf{K}\lambda}}{\partial t^2} = 0 \Rightarrow k^2 A_{\mathbf{K}\lambda}(t) - \frac{1}{c^2} \frac{\partial^2 A_{\mathbf{K}\lambda}(t)}{\partial t^2} = 0$$

and as $\omega_k = ck$ (ω_k , being the modal angular frequency)

$$\frac{\partial^2 A_{\mathbf{K}\lambda}(t)}{\partial t^2} + \omega_k^2 A_{\mathbf{K}\lambda}(t) = 0$$

and it is seen therefore that a field mode is just a simple harmonic oscillator of angular frequency ω_k . Solutions of the above are of the form

$$A_{\mathbf{K}\lambda}(t) = A_{\mathbf{K}\lambda} e^{-i\omega_k t}$$

but we have seen that

$$\mathbf{A}(\mathbf{r}, t) = A_{\mathbf{K}\lambda}(t) e^{i\mathbf{K}\cdot\mathbf{r}} + A_{\mathbf{K}\lambda}^*(t) e^{-i\mathbf{K}\cdot\mathbf{r}},$$

so we get

$$A_{\mathbf{K}\lambda}(\mathbf{r}, t) = A_{\mathbf{K}\lambda} e^{(-i\omega_k t + i\mathbf{K}\cdot\mathbf{r})} + A_{\mathbf{K}\lambda}^* e^{(i\omega_k t - i\mathbf{K}\cdot\mathbf{r})}$$

and the complete vector potential is the sum of the above over all modes and both polarizations, i.e.

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{K}} \sum_{\lambda=1,2} \mathbf{e}_{\mathbf{K}\lambda} A_{\mathbf{K}\lambda}(\mathbf{r}, t) = \sum_{\mathbf{K}} \sum_{\lambda=1,2} \mathbf{e}_{\mathbf{K}\lambda} (A_{\mathbf{K}\lambda} e^{(-i\omega_k t + i\mathbf{K}\cdot\mathbf{r})} + A_{\mathbf{K}\lambda}^* e^{(i\omega_k t - i\mathbf{K}\cdot\mathbf{r})}).$$

The complete (transverse) electric field is obtained from the above by noting $\mathbf{E}_T(\mathbf{r}, t) = -\partial \mathbf{A}(\mathbf{r}, t) / \partial t$ (as in the Coulomb gauge \mathbf{A} is wholly transverse and in the absence of sources $\nabla \phi = 0$ as the charge density = 0). So,

$$\mathbf{E}_T(\mathbf{r}, t) = \sum_{\mathbf{K}} \sum_{\lambda=1,2} \mathbf{e}_{\mathbf{K}\lambda} E_{\mathbf{K}\lambda}(\mathbf{r}, t)$$

where

$$E_{\mathbf{K}\lambda}(\mathbf{r}, t) = i\omega_k \{ A_{\mathbf{K}\lambda} e^{(-i\omega_k t + i\mathbf{K}\cdot\mathbf{r})} - A_{\mathbf{K}\lambda}^* e^{(i\omega_k t - i\mathbf{K}\cdot\mathbf{r})} \}.$$

Similarly from $\mathbf{B} = \nabla \times \mathbf{A}$ we get

$$\mathbf{B}_T(\mathbf{r}, t) = \sum_{\mathbf{K}} \sum_{\lambda=1,2} \frac{\mathbf{K} \times \mathbf{e}_{\mathbf{K}\lambda}}{k} B_{\mathbf{K}\lambda}(\mathbf{r}, t)$$

where

$$B_{\mathbf{K}\lambda}(\mathbf{r}, t) = ik \{ A_{\mathbf{K}\lambda} e^{(-i\omega_k t + i\mathbf{K}\cdot\mathbf{r})} - A_{\mathbf{K}\lambda}^* e^{(i\omega_k t - i\mathbf{K}\cdot\mathbf{r})} \}.$$

(Note - the ratio of the two is $\omega_k/k = c$ as we would hope!)

The total energy of the e-m radiation is given by the integral over the volume of the cube of the intensities of the E and B fields and is given by

$$\mathcal{E}_R = \frac{1}{2} \int_{Vol} [\epsilon_0 \mathbf{E}_T(\mathbf{r}, t) \cdot \mathbf{E}_T(\mathbf{r}, t) + \mu_0^{-1} \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{B}(\mathbf{r}, t)] dV$$

Substituting in our expressions for $\mathbf{E}_T(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ (see [37] page 133) in terms of the vector potential and simplifying, by noting that the time varying terms from the magnetic and electric components cancel (just as they would in a real LC tuned circuit where the total energy would be constant and energy would be stored alternately as magnetic energy in the inductor, $\frac{1}{2}LI^2$ and electric energy in the capacitor, $\frac{1}{2}CV^2$), the total energy in terms of the vector potential comes out as

$$\mathcal{E}_R = \sum_{\mathbf{K}} \sum_{\lambda=1,2} \mathcal{E}_{K\lambda}$$

with

$$\mathcal{E}_{K\lambda} = \epsilon_0 V \omega_k^2 (A_{K\lambda} A_{K\lambda}^* + A_{K\lambda}^* A_{K\lambda}).$$

Rather than combine terms we will leave them as above, as we will arrive at quantum operators in a similar form and will be able to associate the classical vector potentials and their corresponding quantum operators.

10.6 Quantization of the electromagnetic field

The electromagnetic field is quantized by associating a quantum mechanical harmonic oscillator with each mode of the electromagnetic radiation field in the quantization cavity of the previous section. Here we mainly follow the treatment in Loudon [37].

The modes to which the quantum mechanical operators refer are indicated by \mathbf{K}, λ subscripts where \mathbf{K} is the wave vector for a wave of angular frequency ω_k and λ takes the value 1 or 2 depending on the polarization state.

The creation and destruction operators for cavity mode $\mathbf{K}\lambda$ are

$$\hat{a}_{K\lambda} |n_{K\lambda}\rangle = n_{K\lambda}^{1/2} |n_{K\lambda} - 1\rangle \text{ and } \hat{a}_{K\lambda}^\dagger |n_{K\lambda}\rangle = (n_{K\lambda} + 1)^{1/2} |n_{K\lambda} + 1\rangle$$

(note the eigenvalues above correspond to the note on normalization in section 10.3)

The physical interpretation is that these operators destroy and create one photon of energy $\hbar\omega_k$ in mode $\mathbf{K}\lambda$. The number of photons in a given mode is

$$\hat{n}_{\mathbf{K}\lambda} = \hat{a}_{\mathbf{K}\lambda}^\dagger \hat{a}_{\mathbf{K}\lambda}$$

with the eigenvalue relation

$$\hat{n}_{\mathbf{K}\lambda} |n_{\mathbf{K}\lambda}\rangle = \hat{a}_{\mathbf{K}\lambda}^\dagger \hat{a}_{\mathbf{K}\lambda} |n_{\mathbf{K}\lambda}\rangle = n_{\mathbf{K}\lambda} |n_{\mathbf{K}\lambda}\rangle, \quad n_{\mathbf{K}\lambda} = 0, 1, 2, \dots$$

The states $|n_{\mathbf{K}\lambda}\rangle$ are known as photon number states or “Fock” states of the electromagnetic field.

A number state for the total electromagnetic field in the cavity is specified by a string of photon numbers, one for each of the allowed modes. The different cavity modes are independent (we saw this in the classical case and it is as a result of the linearity of the Maxwell equations), and as a result the associated operators commute and just as for the single harmonic oscillator we had

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

we now get the equivalent

$$[\hat{a}_{\mathbf{K}\lambda}, \hat{a}_{\mathbf{K}'\lambda'}^\dagger] = \delta_{\mathbf{K},\mathbf{K}'} \delta_{\lambda,\lambda'}.$$

The state of the total field is a product of the states of the individual modes

$$|n_{\mathbf{K}_{11}}, n_{\mathbf{K}_{12}}, n_{\mathbf{K}_{21}}, n_{\mathbf{K}_{22}}, \dots\rangle = |n_{\mathbf{K}_{11}}\rangle |n_{\mathbf{K}_{12}}\rangle |n_{\mathbf{K}_{21}}\rangle |n_{\mathbf{K}_{22}}\rangle = |\{n_{\mathbf{K}\lambda}\}\rangle$$

where $\{n_{\mathbf{K}\lambda}\}$ denotes the complete set of numbers that specify the excitation levels (number of photons) in all the harmonic oscillators associated with the cavity modes.

The Hamiltonian for the radiation field as a whole is the sum of the contributions from each mode. We saw that for the harmonic oscillator,

$$\hat{H} = \frac{1}{2} \hbar\omega (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}).$$

Now we have

$$\hat{H}_{\mathbf{K}\lambda} = \frac{1}{2}\hbar\omega_k(\hat{a}_{\mathbf{K}\lambda}\hat{a}_{\mathbf{K}\lambda}^\dagger + \hat{a}_{\mathbf{K}\lambda}^\dagger\hat{a}_{\mathbf{K}\lambda})$$

and

$$\hat{H}_R = \sum_{\mathbf{K}} \sum_{\lambda} \hat{H}_{\mathbf{K}\lambda}.$$

A comparison of

$$\hat{H}_{\mathbf{K}\lambda} = \frac{1}{2}\hbar\omega_k(\hat{a}_{\mathbf{K}\lambda}\hat{a}_{\mathbf{K}\lambda}^\dagger + \hat{a}_{\mathbf{K}\lambda}^\dagger\hat{a}_{\mathbf{K}\lambda})$$

with

$$\mathcal{E}_{\mathbf{K}\lambda} = \varepsilon_0 V \omega_k^2 (A_{\mathbf{K}\lambda} A_{\mathbf{K}\lambda}^* + A_{\mathbf{K}\lambda}^* A_{\mathbf{K}\lambda})$$

(from the classical theory) suggest that the quantum mechanical vector potential operators should be

$$A_{\mathbf{K}\lambda} \rightarrow (\hbar/2\varepsilon_0 V \omega_k)^{1/2} \hat{a}_{\mathbf{K}\lambda}$$

and

$$A_{\mathbf{K}\lambda}^* \rightarrow (\hbar/2\varepsilon_0 V \omega_k)^{1/2} \hat{a}_{\mathbf{K}\lambda}^\dagger.$$

From

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{K}} \sum_{\lambda=1,2} \mathbf{e}_{\mathbf{K}\lambda} A_{\mathbf{K}\lambda}(\mathbf{r}, t) = \sum_{\mathbf{K}} \sum_{\lambda=1,2} \mathbf{e}_{\mathbf{K}\lambda} (A_{\mathbf{K}\lambda} e^{(-i\omega_k t + i\mathbf{K} \cdot \mathbf{r})} + A_{\mathbf{K}\lambda}^* e^{(i\omega_k t - i\mathbf{K} \cdot \mathbf{r})})$$

we now get

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_{\mathbf{K}} \sum_{\lambda=1,2} \mathbf{e}_{\mathbf{K}\lambda} \hat{A}_{\mathbf{K}\lambda}(\mathbf{r}, t)$$

where

$$\hat{A}_{\mathbf{K}\lambda}(\mathbf{r}, t) = (\hbar/2\varepsilon_0 V \omega_k)^{1/2} (\hat{a}_{\mathbf{K}\lambda} e^{(-i\omega_k t + i\mathbf{K} \cdot \mathbf{r})} + \hat{a}_{\mathbf{K}\lambda}^\dagger e^{(i\omega_k t - i\mathbf{K} \cdot \mathbf{r})}).$$

We have now quantized the field. Using the above and the classical relationships between the electric field and the vector potential and the magnetic field and the vector potential, it is also possible to generate electric field and magnetic field quantum operators.

Whereas the total energy in the classical field was

$$\mathcal{E}_R = \sum_{\mathbf{K}} \sum_{\lambda} \mathcal{E}_{\mathbf{K}\lambda} = \sum_{\mathbf{K}} \sum_{\lambda} \varepsilon_0 V \omega_k^2 (A_{\mathbf{K}\lambda} A_{\mathbf{K}\lambda}^* + A_{\mathbf{K}\lambda}^* A_{\mathbf{K}\lambda}),$$

we now have

$$\hat{H}_R = \sum_{\mathbf{K}} \sum_{\lambda} \hat{H}_{\mathbf{K}\lambda} = \sum_{\mathbf{K}} \sum_{\lambda} \frac{1}{2} \hbar \omega_k (\hat{a}_{\mathbf{K}\lambda} \hat{a}_{\mathbf{K}\lambda}^\dagger + \hat{a}_{\mathbf{K}\lambda}^\dagger \hat{a}_{\mathbf{K}\lambda}).$$

We saw in the case of the single harmonic oscillator that

$$\frac{1}{2} \hbar \omega (\hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a}) = \hbar \omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}).$$

Thus

$$\hat{H}_R = \sum_{\mathbf{K}} \sum_{\lambda} \hbar \omega_k (\hat{a}_{\mathbf{K}\lambda}^\dagger \hat{a}_{\mathbf{K}\lambda} + \frac{1}{2})$$

and just as $\hat{H}|n\rangle = \hbar \omega (\hat{a}^\dagger \hat{a} + \frac{1}{2})|n\rangle = E_n|n\rangle$ with $E_n|n\rangle = \hbar \omega (n + \frac{1}{2})|n\rangle$ in the single oscillator case, the energy eigenvalue relation for the multimode number state is

$$\hat{H}_R|\{n_{\mathbf{K}\lambda}\}\rangle = \sum_{\mathbf{K}} \sum_{\lambda} \hbar \omega_k (n_{\mathbf{K}\lambda} + \frac{1}{2})|\{n_{\mathbf{K}\lambda}\}\rangle.$$

The ground state of the electromagnetic field (i.e. when no photons are excited) is the state when $n_{\mathbf{K}\lambda} = 0$ for all \mathbf{K} and λ and $\hat{a}_{\mathbf{K}\lambda}|\{0\}\rangle = 0$ for all \mathbf{K} and λ . The eigenvalue for the zero-point state is thus

$$\hat{H}_R|\{0\}\rangle = \frac{1}{2} \sum_{\mathbf{K}} \sum_{\lambda} \hbar \omega_k |\{0\}\rangle \equiv E_0|\{0\}\rangle$$

where the ground state energy

$$E_0 = \frac{1}{2} \sum_{\mathbf{K}} \sum_{\lambda} \hbar \omega_k = \sum_{\mathbf{K}} \hbar \omega_k.$$

We can rewrite the eigenvalue relation, separating out the energy in excited states from the zero-point energy, thus

$$\hat{H}_R|\{n_{\mathbf{K}\lambda}\}\rangle = (E_R + E_0)|\{n_{\mathbf{K}\lambda}\}\rangle$$

where

$$E_R = \sum_{\mathbf{K}} \sum_{\lambda} \hbar \omega_k n_{\mathbf{K}\lambda}.$$

This might be the energy in the light or other radiation from an astronomical source such as a star or galaxy, that we would expect a distant observer to see as “cosmologically redshifted”.

10.7 The quantum field in the new metric

The analysis in the last section tacitly assumes the Minkowski space-time i.e. $g_{\mu\nu} = \eta_{\mu\nu}$. This comes about from the start as Maxwell's equations, from which we show that a field mode is equivalent to a harmonic oscillator, are Lorentz invariant. We will now look at these quantized field modes in an expanding universe, in our new time coordinate.

In a similar argument to that used with the single harmonic oscillator, where we wrote $E_n = (n + \frac{1}{2})\hbar\omega'_{\text{emit}}$, $n = 0, 1, 2, \dots$ where $\omega'_{\text{emit}} = a\omega_{\text{emit}} = (\Phi_C + 1)\omega_{\text{emit}} = \omega_{\text{obs}}$, We can now write

$$E_R = \sum_{\mathbf{K}} \sum_{\lambda} \hbar(\omega_k)'_{\text{emit}} n_{\mathbf{K}\lambda}$$

where

$$(\omega_k)'_{\text{emit}} = a(\omega_k)_{\text{emit}} = (\Phi_C + 1)(\omega_k)_{\text{emit}} = (\omega_k)_{\text{obs}}.$$

Thus the energy as seen by the observer is

$$E_R = \sum_{\mathbf{K}} \sum_{\lambda} \hbar(\Phi_C + 1)(\omega_k)_{\text{emit}} n_{\mathbf{K}\lambda}.$$

Again it is clear from the above that the theory developed for the quantum field, correctly describes the field in an expanding universe if we replace ω everywhere it occurs with $\omega' = a\omega = (\Phi_c + 1)\omega$ and λ everywhere it occurs with $\lambda' = \lambda/a = \lambda/(\Phi_c + 1)$. This is how cosmological redshift is incorporated into a quantum field view of the radiation from distant astronomical sources.

10.8 Vacuum energy

Of course we should also consider the zero-point or vacuum energy, where we had

$$E_0 = \sum_{\mathbf{K}} \hbar\omega_k.$$

we must now write

$$E_0 = \sum_{\mathbf{K}} \hbar(\Phi_C + 1)\omega_k.$$

or

$$E_0 = \sum_{\mathbf{K}} \hbar\omega'_k.$$

where $\omega' = a\omega = (\Phi_c + 1)\omega$.

Chapter 11

Conclusions and further thoughts

11.1 Cosmological redshift and quantum theory

We compared cosmological redshift to the gravitational redshift as measured by Pound and Rebka. If we observe a source of radiation at the earth's surface from the top of a tall tower we see the radiation at a lower frequency than we would if the source were at the top of the tower with us. This is gravitational redshift and we could say that it is due to the gravitational time dilation caused by the difference in gravitational potential between the source and the observer.

In the same way we have suggested that the redshift of light from distant sources is the result of time dilation, due to the difference in a “cosmic” potential between the time of emission and time of observation. We have pointed out that cosmological redshift can't be the result of interactions of the radiation field with other fields/particles and is not describable in the standard quantum theory when the time coordinate is that of the Robertson Walker metric. We have introduced a new metric (including a new time coordinate) and have reformulated the field theory such that cosmological redshift is a natural outcome of the theory without the need for photons to undergo interactions that would blur spectral lines and images.

It is worth noting that photon number is conserved in this theory. This is not necessarily the case for quantum field theories in curved space-times. Robert Wald in “Quantum Field theory in Curved Space-time and Black Hole Thermodynamics” says *“Although in appropriate circumstances a particle interpretation of the theory may be available, the notion of “particles” plays no fundamental role either in the formulation or interpretation of the theory.”* and *“In curved space-times which are asymptotically stationary in the past and/or future, natural particle interpretations are also available in these asymptotic regions. However, in other circumstances, the notion of “particle” is, at best, of very limited utility”* [42]. In our case there is no problem with a particle interpretation for co-moving observers and for such observers, photon number (of the background radiation field) is essentially constant.

11.2 Notes on dynamics and time dilation

We have already looked at the dynamics of spatially flat Robertson Walker universe that has $\Lambda = 0$. We know that

$$\dot{R}^2(t) + 2R(t)\ddot{R}(t) = 0.$$

This equation governs the dynamics of the Universe (see figure 4.3) and must hold in a low pressure universe with zero cosmological constant, that is described by the spatially flat Robertson Walker metric.

We will plot $\frac{dn}{d\tau}$ against Φ_C .

We know that $a = (\Phi_C + 1)$, so $\frac{dn}{d\tau} = \frac{1}{\Phi_C + 1}$ so plotting $\frac{dn}{d\tau}$ against Φ_C for Φ_C varying from -1 to 0, we get the plot shown in figure 12.1. The vertical axis shows the ratio of dn to dt and the horizontal shows the potential Φ_C .

From this we can see that in terms of the new time coordinate, n , the Big Bang happened in the infinite past. This is not a new dynamics of the universe, it is just looking at the old dynamics in a new time coordinate. The same graph holds regardless of value of Λ , so the same can be said of

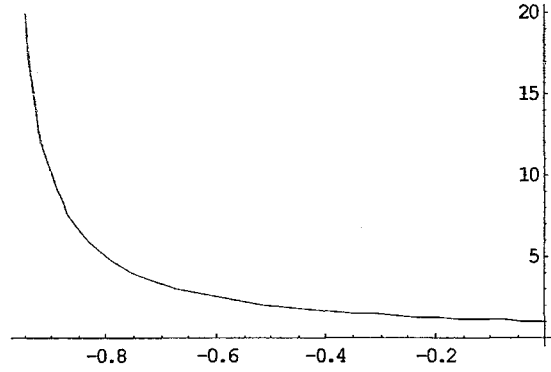


Figure 11.1 Ratio of dn to dt plotted against the Potential Φ_C

the dynamics of the universe with a non zero cosmological constant. Seen in terms of “time now”, time passed more slowly in the early stages of the Universe.

11.3 Further work on massive particles with peculiar velocity with respect to the local rest frame.

It would be interesting to revisit the massive particle case and look at how the velocity changes can be described in the new time coordinate. It would also lend weight to the concept of the global cosmic potential if it could be shown how the change in local kinetic energy of a particle as it slows and becomes co-moving is related to its mass time the change in cosmic potential. In particular, further work could be carried out to calculate the possible combinations of initial energies and distance travelled, of cosmic rays given an observed energy on arrival, using calculations such as are shown in Figure 5. Such combinations might provide insight into the source of cosmic rays of various energies.

11.4 On the observability of the increasing potential Φ_C when $\dot{a}(=H)$ is constant

In much the same way as we might perform a “Pound Rebka” like experiment comparing the frequencies of a light beam from a source at the bottom of a tall tower with measurements made at the top and bottom of the tower, it is tempting to wonder if a similar shift in frequency might be measured in an experiment interfering light directly from a laser source with light from the same source that had been delayed by perhaps sending it on a long journey to a remote mirror and back (A to B and back in Fig 12.2). In this way we would be comparing the frequency of the source “now” with what it was some short time ago when the potential was (slightly) less. The slowly accumulating phase difference due to the shift in frequency would cause the interference fringes to drift (slowly!) across the screen. Fig 12.2 shows a possible arrangement. A real experiment would use apparatus set up on the Earth. However, to analyze what might be observed in such an experiment, we will imagine that, in an expanding universe of uniform comoving dust, we are able to set up an observer with a laser light source and some distance away a, mirror, both observer with laser and the mirror being comoving, a fixed coordinate distance apart. We will ignore here the difficulties of upsetting the homogeneity and isotropy by the introduction of the observer and the apparatus perhaps by assuming that the observers mass and the apparatus mass are zero. We would expect the observer to record that the reflected light is redshifted as if it were from a source twice the coordinate distance away from the observer as that of the mirror since the change in scale factor or potential, however we view it, during the time of flight of a photon, would be the same in both cases. In the real Universe, we can’t expect to be able to set up truly comoving observers and mirrors and so we ask the question “If in a expanding universe of uniform dust, we set up the same observer and apparatus as before but make the distance between the observer and mirror a fixed proper distance, will any redshift be observed?” Both experiments are considered below and we will assume that any real experiment carried

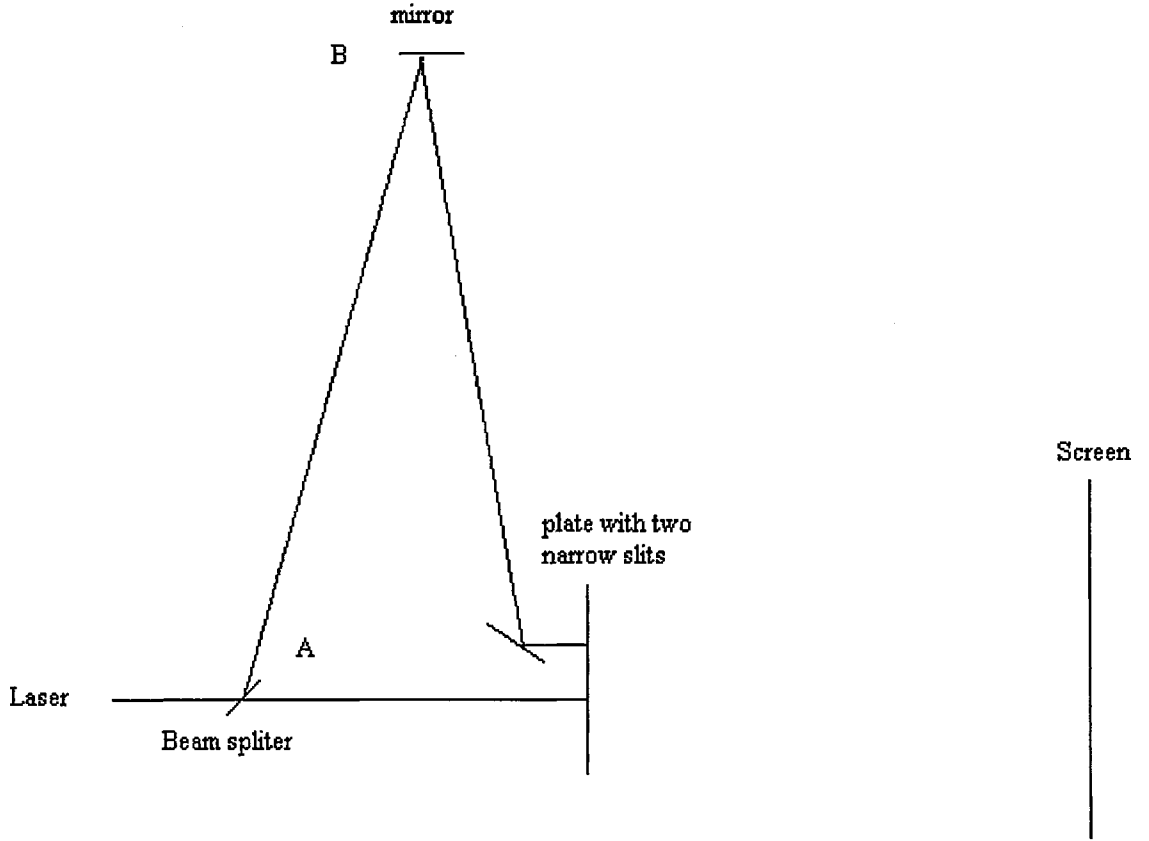


Figure 11.2 Experimental apparatus for detecting rising potential

out on Earth would produce similar results to the fixed proper distance case.

If, in a uniform expanding universe, the main apparatus (laser, beam splitter, plate and screen) were co-moving and the remote mirror at B were also co-moving, a proper distance r away from the main apparatus at the instant of reflection, then we know that the frequency shift would be give by

$$\frac{f_{\text{reflected}}}{f_{\text{emitted}}} = a = 1 - 2\dot{a}\frac{r}{c}.$$

If, however, the mirror is a fixed proper distance from the main apparatus then, if the main apparatus is co-moving, the mirror at B has a velocity towards A (with respect to the local co-moving frame) of $\dot{a}r/c$ and would blueshift the frequency by a factor $(1 + \dot{a}r/c)^2 \simeq (1 + 2\dot{a}r/c)$ for small r .

The total effect would therefore be

$$\frac{f_{\text{reflected}}}{f_{\text{emitted}}} \simeq (1 - 2\dot{a}\frac{r}{c})(1 + 2\dot{a}\frac{r}{c})$$

which, for small r is

$$\frac{f_{\text{reflected}}}{f_{\text{emitted}}} \simeq 1. \quad (11.1)$$

In the above analysis we have assumed that \dot{a} is constant. In this case, we see that there will be no overall redshift, for any practical experiment of this sort and the effects of the rising potential are unobservable.

This is just as well! If, instead of using laser light, we were looking at the image of a clock face, directly and via the reflected image from the mirror at B, then the equivalent of an accumulating phase difference would be an increasing difference between the times shown in the two clock images. This is the same as saying that the delay time for the “round trip”, A to B and back to A is increasing, or equivalently that the locally measured speed of light is decreasing. Thus the speed of light, c and quantities that depend on it such as ϵ_0 and perhaps, the fine structure constant, would depend strongly on the value of \dot{a} . A strong dependence on \dot{a} would have been observed by now if the effects of such a dependence on the chemistry of the earlier universe permitted beings such as ourselves to be here at all. There remains the possibility that observable effects may occur if the scale of the experiment is very large and if \dot{a} is not constant.

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